On the Pfaffian Property of Annihilating Random Walk and Coalescing Random Walk

by

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Contents

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Declarations

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. It has been composed by myself and has not been submitted in any previous application for any degree

Abstract

In this thesis we are to investigate two discrete interacting particle systems, namely annihilating random walk and coalescing random walk. By mapping the annihilating random walk to Glauber model and employing empty interval method respectively, we prove there is a similar structure behind them albeit their apparent differences, that is, they are both Pfaffian point process under a special initial condition.

Then we extend the result to investigate whether the Pfaffian property preserves in the case of multi-time correlation function, which is called extended Pfaffian property. And we also investigate the case which the initial condition is changed from independent particles to another peculiar one-sided initial condition and proved it also preserved the Pfaffian property.

Chapter 1

Introduction

In this thesis we are to investigate two important interacting particle system: annihilating random walk (ARW) and coalescing random walk (CRW). We confine our attention to one-dimensional case only at the moment.

In both systems there are particles moving to left or right on a discrete lattice at some rate. Each site can hold at most one particle. When a particle move to an occupied site, they annihilate each other in the case of ARW or merge into one particle in the case of CRW. In our research the reaction rate is assumed to be infinite.

Although their interactions seem quite different, these two systems has been observed to be related in some peculiar ways. For example, Arratia [16] proved the thinning relation for the large time asymptotic of the particle density. And Brunet and ben-Avraham[9] explained the precise meaning of similarity of the two systems in terms of the hierarchies of multiple-pooint correlation functions. Basically the results show that for nice enough initial conditions, if we remove half of the particles in CRW at random, we can obtain an ARW.

Besides the similarity between themselves, these two systems have been observed to be related to other interacting particle systems as well.

Annihilating random walk has been observed to be related to a dynamics Ising model, which is called Glauber model, for a long time.

Ising model was developed as a model of statistical mechanics to explain ferromagnetic behaviour in matter. One of the remarkable results is that the system has no phase change in 1D case but there is phase change in 2D case. Since then there were a lot of generalisation of Ising model, such as XY model, Pott model.

As a model of equilibrium statistical mechanics, Ising model is very interesting but one would like to study how a system approaches the equilibrium state. Glauber

[8] developed the dynamic spin model and it was proved that the equilibrium state of a 1D Glauber model is exactly Ising model. After then, many people studied the model and made other generalisations. For example, Godreche [13] [14] generalised the Glauber model to the asymmetric case.

As both the Glauber model and annihilating random walk(ARW) are two-state systems, people explored methods to study ARW by mapping it to the Glauber model. One can define an object called domain wall, which stays in the dual discrete space between the spins, as follows.

If two neighbouring spins are of different directions we say that they is a domain wall between them. If the spins are of same direction we say there is no domain wall between them. For example, consider the spin configuration 1 1 -1. The corresponding domain walls are 0 1. Now if the spin in the middle flips to -1, the domain walls will become 1 0. So the flipping of the spins corresponds to the movement of domain walls. If there is a spin configuration 1 -1 1 and the middle spin flips to 1, the corresponding domain walls will change from 1 1 to 0 0. Therefore we can use the definition of domain walls to establish a mapping from spin chain to ARW.

Schutz [15] studied the particle density of ARW using free fermionic representation, which is a powerful tool we also employed in this thesis. He investigated the time dependence of particle density for product measure initial condition and step-function initial condition and also the large time asymptotic behaviour. He also applied Arratia's result to obtain the particle density for CRW, which was used to simulate the behaviour of excitons.

On the other hand, coalescing random walk(CRW) is related to another interacting particle system called voter model. In fact voter model is the dual process of CRW and thus one has been used to study another. For example Bramson and Griffeath [22] studied the asymptotics for the particle density for voter model starting with one particle at the origin and CRW with the whole Z^d as initial condition. Another useful method to study CRW, which we will employ in this thesis, is the empty interval probability developed by Ben Avraham [9].

Although lots of studies have been done on the particle density of these interacting particle systems, few have attempted to further study the particle correlation function. Following the approach of Glauber, we can see that the particle correlation function is an object of importance because the reduced probability can be expressed as a summation of particle correlation. Therefore we would like to investigate its functional behaviour.

For some particle systems the particle correlation function is a determinant of kernel functions which are thus called determinantal point processes [6]. Examples include particle correlation function of fermionic gas and joint distribution of eigenvalues of certain ensembles of random matrices. In the investigation of random matrices it turns out there are other generalisations of determinantal point processes. One of them is Pfaffian point process in which the correlation is a Pfaffian (square root of a determinant of a $2n \times 2n$ anti-symmetric matrix). For example, $\beta = 1$ and $\beta = 4$ polynomial ensembles of random matrices are Pfaffian point processes.

As Pfaffian point process is more general than determinantal process, since a determinant can always be expressed as a Pfaffian, people started to look for these systems. Katori [12] demonstrated that a determinantal process starting from orthogonal symmetry initial condition is a Pfaffian point process. Furthermore, recent researches also showed that Ginibre enembles are related to Pfaffian point process[24] [3].

In this thesis, we are going to show that ARW and CRW are Pfaffian point process under two initial conditions: maximal entrance law and one-sided function. We will also consider some variations of the systems, including spontaneous creation of particles, asymmetric cases and position-dependent random walks.

Another important result is that ARW and CRW possess extended Pfaffian property, which means the multi-time correlation is also a Pfaffain point process. We will prove that in the case of the most general position-dependent random walk ARW and CRW possess extended Pfaffian property under maximal entrance law.

1.1 Pfaffian

The determinant of an anti-symmetric matrix A, i.e. $A_{i,j} = A_{j,i}$, is a square of a polynomial of the entries and therefore it is natural to define the square root of the determinant which is called Pfaffian.

The definition of Pfaffian is given below:

Definition 1. *Given a $2n \times 2n$ skew-symmetric matrix A, the Pfaffian $Pf(A)$ of A is the square root of the determinant of A defined by*

$$Pf(A) = \sum_{\sigma \in \sum_{2n}} sgn\,(\sigma)\, a_{i_1,j_1} a_{i_2,j_2} \dots a_{i_n,j_n}$$

where $\sum_{2n}$ is the summation over all the permutations σ of $\{1, 2, \dots, 2n\}$ given by $\sigma\,(2k-1) = i_k$, $\sigma\,(2k) = j_k$ for $k = 1, \dots, n$. The permutations have to satisfy two conditions: $i_k < j_k$ for all k and $i_1 < i_2 < \dots < i_n$.

For example, if $A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$, $\mathrm{Pf}(A) = a$. If $A = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix}$, $\mathrm{Pf}(A) = af - be + cd$.

A convenient way to manipulate Pfaffian to obtain useful identities is by using superintegral. A good reference is [21].

A useful formula for the decomposition of Pfaffians is the following:

Lemma 1. *For two $2n \times 2n$ matrices A and B [1],*

$$Pf(A+B) = \sum_J (-1)^{|J|/2} (-1)^{s(J)} Pf(A|_J)\, Pf(B|_{J^c})$$

where the sum is over all subsets $J \subseteq \{1, 2, \ldots, 2n\}$ with an even number of terms; $J^c = \{1, 2, \ldots, 2n\} \backslash J$; $s(J) = \sum_{j \in J} j$ (and $s(\emptyset) = 0$); and where $A|_J$ means the submatrix of A formed by the rows and columns indexed by elements of J (and the Pfaffian of the empty matrix is taken to have value 1).

Another lemma we will use later is the following, the proof of which will not be given here:

Lemma 2. *For a skew-symmetric $2n \times 2x$ matrix A,*

$$Pf\left(A^T\right) = (-1)^n \, Pf(A)\,. \tag{1.1}$$

1.2 Pfaffian point process

1.2.1 Point process

Albeit its name, in general a point process is not really a time dependent process. Here we will only give a basic definition of point process. For details one can refer to the book by Daley, D.J., Vere-Jones, D [5].

For a locally compact second countable Hausdorff space S, a point process is a map ξ which maps a bounded subset A/inS to locally finite counting measures and can be written as :

$$\xi(A) = \sum_{i=1}^{N} \delta_{X_i}$$

where X_i are the random positions. If $X_i \neq X_j$ whenever $i \neq j$, then we say that the point process is simple.

Therefore a point process can be viewed as putting particles randomly on a space and hence at every time t a particle system can be regarded as a simple point process.

1.2.2 Pfaffian point process

Since in our research the particles are on a discrete lattice and the process is simple, our correlation function at time t has a simple form:

$$\rho_t^n\left(x_1, \ldots, x_n\right) = \mathrm{E}_t\left(\prod_{i=1}^{n} n_{x_i}\right)$$

where $n_{x_i} \in \{0, 1\}$ is the occupation number at stie x_i.For the continuous case the definition of correlation function can be a bit technical and the interested readers can refer to Daley, D.J., Vere-Jones, D [5].

Definition 2. *A random point process is called Pfaffian if its point correlation functions have a Pfaffian form[6], i.e.*

$$\rho_n\left(x_1, \ldots, x_n; t\right) = Pf\left(K\left(x_i, x_j\right)\right)_{i,j=1,\ldots,n}, \; n \geq 1$$

where $K(x, y) = \begin{pmatrix} K_{11}(x, y) & K_{12}(x, y) \\ K_{21}(x, y) & K_{22}(x, y) \end{pmatrix}$ *is called the matrix kernel which is subject to the constraint* $K_{ij}(x, y) = -K_{ji}(y, x)$ *in order to make the* $2n \times 2n$ *matrix skew-symmetric.*

The term $K(x_i, x_j)$ can be thought of as the element K_{ij} of a $n \times n$ matrix K and is a 2×2 matrix whose elements are functions of the positions x_i, x_j.

Chapter 2

Glauber model and annihilating random walk(ARW)

To investigate the system of annihilating random walk(ARW), my approach is to start from studying the behaviour of Glauber model by employing the free Fermonic formulation and then map to the system of annihilating random walk. The object in which we are interested is the spin correlation function $E_t\left(\prod_{i=1}^{n} s_{x_i}\right)$, which can be derived from the generating function $G_t\left(\vec{J}\right)$. We can thus find the generating function for ARW by mapping this expression to that of ARW.

2.1 Glauber model

2.1.1 Introduction to Glauber model

On an infinite discrete one-dimensional lattice Z, every site x is occupied by a spin $s_x \in \{-1, +1\}$. Every spin can flip depending of the status of its nearest neighbours. In other words, the transition rate ω of the spin s_k at position k is

$$\omega\left(s_{k-1}, s_k, s_{k+1}\right) = 1 + \gamma s_k\left(s_{k-1} + s_{k+1}\right),$$

where $\gamma = \frac{-1}{2}\tanh\left(\frac{2J}{kT}\right)$ and J is the interaction between the spins and T is the temperature. Therefore γ represents the "temperature" of the system. At zero temperature, where $\gamma = \frac{-1}{2}$, the spins tend to align with each other so the transition rate for spin s_k at position k should be zero when its neighbours are aligned with it and no spontaneous flipping will happen. In this chapter we only consider the case of zero temperature.

Let $\vec{s}$ denote the spin configuration on the lattice, ideally we want to find the probability $P_t(\vec{s})$ that the spin configuration is $\vec{s}$. We have a master equation for this probability:

$$\partial_t P_t(\vec{s}) = \sum_k \left[\omega(s_{k-1}, -s_k, s_{k+1}) P(\ldots s_{k-1}, -s_k, s_{k+1}\ldots) - \omega(s_{k-1}, s_k, s_{k+1}) P(\vec{s})\right] \tag{2.1}$$

However, this probability contains more information than we actually need and is difficult to calculate. So instead we investigate the reduced probability $P_t(s_{x_1}, \ldots, s_{x_n})$, which is the probability that the n spins s_{x_i} are in the configuration $\{s_{x_1}, \ldots, s_{x_n}\}$. For the convenience of calculation, we also confine our attention to a chain of N spins but we can always make N large enough to contain all the n spins we are interested in.

We can obtain the reduced probability $P_t(s_{x_1}, \ldots, s_{x_n})$ from spin correlations $\mathrm{E}_t(s_{x_1} \cdots s_{x_n})$. Consider the function $\frac{1}{2}\left(1 + s_{x_i} s'_{x_i}\right)$, where $s_{x_i}, s'_{x_i} \in \{-1, 1\}$. It can be seen that

$$\frac{1}{2}\left(1 + s_{x_i} s'_{x_i}\right) = \begin{cases} 1 & \text{if } s_{x_i} = s'_{x_i} \\ 0 & \text{if } s_{x_i} = -s'_{x_i} \end{cases}.$$

Therefore we can expand the reduced probability $P_t(s_{x_1}, \ldots, s_{x_n})$ by

$$\begin{aligned} & P_t(s_{x_1}, \ldots, s_{x_n}) \\ = & \frac{1}{2^n} \sum_{\{\vec{s'}\}} \left(1 + s_{x_1} s'_{x_1}\right) \cdots \left(1 + s_{x_n} s'_{x_n}\right) P_t\left(\vec{s'}\right) \\ = & \frac{1}{2^n} \left\{ 1 + \sum_{i=1}^{n} s_{x_i} \left(\sum_{\{\vec{s'}\}} s'_{x_i} P_t\left(\vec{s'}\right) \right) + \sum_{\substack{i,k=1 \\ i \neq k}}^{n} s_{x_i} s_{x_k} \left(\sum_{\{\vec{s'}\}} s'_{x_i} s'_{x_k} P_t\left(\vec{s'}\right) \right) + \cdots \right\} \\ = & \frac{1}{2^n} \left\{ 1 + \sum_{i=1}^{n} s_{x_i} \mathrm{E}_t(s_{x_i}) + \sum_{\substack{i,k=1 \\ i \neq k}}^{n} s_{x_i} s_{x_k} \mathrm{E}_t(s_{x_i} s_{x_k}) + \cdots \right\} \end{aligned}$$

where the factor $\frac{1}{2^N}$ comes from the fact that we have N spins in the chain of spins $\vec{s}$ and hence 2^N spin configurations. Therefore, the reduced probability $P_t(s_{x_1}, \ldots, s_{x_n})$ can be expanded as a summation of products of spins and spin correlations $\mathrm{E}_t(s_{x_1} \cdots s_{x_n})$[8].

The spin correlation $\mathrm{E}_t(s_{x_1} \cdots s_{x_n})$ can be derived from the generating function

$$G_t\left(\vec{J}\right) = \mathrm{E}_t\left(e^{\vec{J} \cdot \vec{s}}\right)$$

by

$$\mathrm{E}_t\left(s_{x_1}\cdots s_{x_n}\right)=\prod_{k=1}^{n}\partial_{J_k}G_t\left(\vec{J}\right)|_{\vec{J}=0}\ .$$

In Glauber's original paper, he calculated only up to 2-point correlation function by employing the technique of ordinary differential equation and involves summation of Bessel functions. However, he mentioned an alternative method which regards the reduced probability as a vector. We are going to follow this approach and introduce the free Fermionic formulation, a computational technique borrowed from quantum field theory, to calculate the generating function $G_t\left(\vec{J}\right)$.

2.1.2 Fermionic treatment of Glauber model

Firstly let us expresses the spins s_{x_i} as unit vectors in a Hilbert space and introduce some useful linear operators.

Let $|s_k\rangle$ be a ket vector which expresses the state of the spin at position k. Define $\tau_k^{(3)}$ to be a linear operator such that

$$\tau_k^{(3)}|s_k\rangle=s_k|s_k\rangle$$

where $s_k\in\{\pm1\}$ and also $\langle s_k|$ be a bra vector such that

$$\langle s_k|\tau_k^{(3)}=\langle s_k|s_k.$$

To describe the entire spin configuration we can construct a vector $|\vec{s}\rangle$:

$$|\vec{s}\rangle=\bigotimes_{k=1}^{N}|s_k\rangle.$$

We also need two useful operators: the raising operator τ_k^+ and the lowering operator τ_k^-,

$$\begin{aligned}\tau_k^+|-1\rangle=|1\rangle,\ \tau_k^+|1\rangle=0\\ \tau_k^-|-1\rangle=0,\ \tau_k^-|1\rangle=|-1\rangle\end{aligned}$$

and also

$$\begin{aligned}\langle -1|\tau_k^-=\langle 1|,\ \langle 1|\tau_k^-=0\\ \langle -1|\tau_k^+=0,\ \langle 1|\tau_k^+=\langle -1|.\end{aligned}$$

Also,

$$\left(\tau_k^+\right)^2 = \left(\tau_k^-\right)^2 = 0. \tag{2.2}$$

A special vector $|\vec{1}\rangle = \bigotimes_k |1_k\rangle$ is needed as well.

Now we can express the generating function $G_t\left(\vec{J}\right)$ in terms of vectors and linear operators:

$$G_t\left(\vec{J}\right) = \langle\vec{1}|e^{\sum_k \tau_k^+} e^{\sum_k J_k\tau_k^{(3)}}|P_t\rangle. \tag{2.3}$$

where

$$|P_t\rangle = \sum_{\vec{s}} P_t\left(\vec{s}\right)|\vec{s}\rangle.$$

Now we can rewrite the master equation (2.1) in the vector notation:

$$\begin{aligned}\partial_t|P_t\rangle &= \sum_{\vec{s}} \partial_t P_t\left(\vec{s}\right)|\vec{s}\rangle \\ &= \sum_k \left(\tau_k^{(1)} - 1\right)\omega\left(\tau_{k-1}^{(3)}, \tau_k^{(3)}, \tau_{k+1}^{(3)}\right)|P_t\rangle\end{aligned} \tag{2.4}$$

where $\tau_k^{(1)}$ is the spin-flip operator and $\tau_k^{(1)}|s_k\rangle = |-s_k\rangle$. For brevity we can write (2.4) in a more compact form:

$$\partial_t|P_t\rangle = -\mathfrak{L}|P_t\rangle$$

where

$$\mathfrak{L} = \sum_k \left(1 - \tau_x^{(1)}\right)\omega\left(\tau_{k-1}^{(3)}, \tau_k^{(3)}, \tau_{k+1}^{(3)}\right)$$

is the Liouvillian operator. We have a solution to this ODE:

$$|P_t\rangle = e^{-\mathfrak{L}t}|P_0\rangle. \tag{2.5}$$

Combining equations (2.3) and (2.5) we have

$$G_t\left(\vec{J}\right) = \langle\vec{1}|e^{\sum_k \tau_k^+} e^{\sum_k J_k\tau_k^{(3)}} e^{-\mathfrak{L}t}|P_0\rangle. \tag{2.6}$$

$|P_0\rangle$ is the initial condition of the chain of spins. In our work we consider the following initial condition:

$$s_k = \begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } 1-p \end{cases}$$

where all the spins s_k have independent probability measure.

We can express this initial condition in terms of vector:

$$\begin{aligned}
|P_0\rangle &= \prod_{k=1}^{N} \left[p|s_k = 1\rangle + (1-p)\,|s_k = -1\rangle\right] \\
&= \prod_{k=1}^{N} \left[p + (1-p)\,\tau_k^-\right] |\vec{1}\rangle \\
&= e^{\sum_k \left(\frac{1-p}{p}\tau_k^- + \ln(p)\right)} |\vec{1}\rangle
\end{aligned}$$

where we have used the relationship (2.2). Therefore we can rewrite equation (2.6) as

$$G_t\left(\vec{J}\right) = \langle\vec{1}|e^{\sum_k \tau_k^+} e^{\sum_k J_k \tau_k^{(3)}} e^{-\mathfrak{L}t} e^{\sum_k \left(\frac{1-p}{p}\tau_k^- + \ln(p)\right)} |\vec{1}\rangle. \tag{2.7}$$

Suppose the chain of spins have zero magnetisation at time $t = 0$, then $p = \frac{1}{2}$. Equation (2.7) becomes

$$G_t\left(\vec{J}\right) = \frac{1}{2^N}\langle\vec{1}|e^{\sum_k \tau_k^+} e^{\sum_k J_k \tau_k^{(3)}} e^{-\mathfrak{L}t} e^{\sum_k \tau_k^-} |\vec{1}\rangle. \tag{2.8}$$

To simplify equation (2.8) we have to simplify the Liouvillian operator $\mathfrak{L}$ to quadratic form in terms of $\tau_k^{(2)}$ and $\tau_k^{(3)}$ only, which enable us to transform the operators $\tau_k^{(2)}$ and $\tau_k^{(3)}$ to fermionic operators using Wigner-Jordan transformation: annihilating operator ψ_k and creating operators $\psi_k^\dagger$ such that the fermionic operators satisfy anti-commutation relations:

$$\{\psi_k, \psi_l^\dagger\} = \delta_{k,l}$$
$$\{\psi_k, \psi_l\} = \{\psi_k^\dagger, \psi_l^\dagger\} = 0.$$

By the relation

$$\tau^{(i)}\tau^{(j)} = -i\epsilon_{i,j,k}\tau^{(k)}$$

where $i, j, k \in \{1, 2, 3\}$ and $\epsilon_{i,j,k}$ is the Levi-Civita symbol, the Liouvillian operator $\mathfrak{L}$ can be rewritten as

$$\mathfrak{L} = \sum_k \left(\tau_k^{(3)} - i\tau_k^{(2)}\right) \cdot \left(\tau_k^{(3)} + \gamma\left(\tau_{k-1}^{(3)} + \tau_{k+1}^{(3)}\right)\right).$$

Now we have to introduce Jordan-Wigner transformation. Firstly we define

$$b_k = \frac{1}{2}\left(\tau_k^{(3)} + i\tau_k^{(2)}\right) \tag{2.9}$$
$$b_k^\dagger = \frac{1}{2}\left(\tau_k^{(3)} - i\tau_k^{(2)}\right). \tag{2.10}$$

These new operators satisfy the following commutation relations at the same site:

$$b_k^2 = {b_k^\dagger}^2 = 0$$
$$\{b_k, b_k^\dagger\} = 1,$$

and these operators commute at different sites:

$$\left[b_k, b_l^\dagger\right] = [b_k, b_l] = 0,\ k \neq l. \tag{2.11}$$

Now by using these b operators we can define creation operators ψ_k and annihilation operators $\psi_k^\dagger$:

$$\psi_k = e^{i\pi J_{k-1}} b_k \tag{2.12}$$
$$\psi_k^\dagger = e^{-i\pi J_{k-1}} b_k^\dagger \tag{2.13}$$

where

$$J_k = \sum_{p \leq k} b_p^\dagger b_p \tag{2.14}$$

Since we have equation (2.11) we get

$$[J_k, J_l] = 0\ \forall k, l$$

and hence the anti-commutation relations of the ψ operators:

$$\{\psi_k, \psi_l^\dagger\} = \delta_{k,l} \tag{2.15}$$
$$\{\psi_k, \psi_l\} = \{\psi_k^\dagger, \psi_l^\dagger\} = 0. \tag{2.16}$$

Therefore the Liouvillian operator $\mathfrak{L}$ can be expressed in terms of ψ operators as:

$$\mathfrak{L} = 2\sum_k \psi_k^\dagger \psi_k + 2\gamma \sum_k \left(\psi_k^\dagger \psi_{k-1} + \psi_k^\dagger \psi_{k+1} - \psi_k^\dagger \psi_{k-1}^\dagger + \psi_k^\dagger \psi_{k+1}^\dagger\right). \tag{2.17}$$

By defining the left vacuum state $\langle 0|$ by

$$\langle 0| = \frac{1}{2^{N/2}}\langle\vec{1}|e^{\sum_k \tau_k^+}$$

and the right vacuum state $|0\rangle$ by

$$\frac{1}{2^{N/2}}e^{\sum_k \tau_k^-}|\vec{1}\rangle = |0\rangle,$$

the equation (2.8) can be rewritten as

$$G_t\left(\vec{J}\right) = \langle 0|e^{\sum_k J_k\tau_k^{(3)}}e^{-\mathfrak{L}t}|0\rangle. \tag{2.18}$$

By equations (2.9) and (2.10) we have

$$b_k|0\rangle = 0 = \langle 0|b_k^\dagger,\ \forall k \tag{2.19}$$

and hence by the commutation relations (2.11) and the definition of ψ operators (2.12) and (2.13) we get

$$\psi_k|0\rangle = 0 = \langle 0|\psi_k^\dagger,\ \forall k. \tag{2.20}$$

That is the reason why $\langle 0|$ and $|0\rangle$ are called left and right vacuum states respectively. Futhermore, since every term in the Liouvillian operater $\mathfrak{L}$ in (2.17) has a $\psi^\dagger$ operater on the left hand side, by (2.20) we have

$$\langle 0|e^{\mathfrak{L}t} = 0. \tag{2.21}$$

From (2.9) and (2.10) we know that $\tau_k^{(3)} = b_k + b_k^\dagger$. Since from (2.11) we know that b_k and $b_k^\dagger$ commute at different sites, we have

$$G_t(\vec{J}) = \langle 0|\prod_k e^{J_k(b_k+b_k^\dagger)}e^{-\mathfrak{L}t}|0\rangle.$$

From (2.12) and (2.13) we have

$$b_k = e^{-i\pi\hat{J}_{k-1}}\psi_k \qquad \text{and} \qquad b_k^\dagger = e^{i\pi\hat{J}_{k-1}}\psi_k,$$

where $\hat{J}_{k-1} = \sum_{p\leq k-1}\psi_p^\dagger\psi_p$. So we can rewrite the $G_t\left(\vec{J}\right)$ in terms of $\psi_k$and $\psi_k^\dagger$, which is

$$G_t(\vec{J}) = \langle 0|\prod_k e^{J_k(e^{-i\pi\hat{J}_{k-1}}\psi_k+e^{i\pi\hat{J}_{k-1}}\psi_k^\dagger)}e^{-\mathfrak{L}t}|0\rangle,$$

where the order in the product is

$$\prod_k e^{J_k(e^{-i\pi\hat{J}_{k-1}}\psi_k+e^{i\pi\hat{J}_{k-1}}\psi_k^\dagger)} = \cdots e^{J_1(e^{-i\pi\hat{J}_0}\psi_1+e^{i\pi\hat{J}_0}\psi_1^\dagger)}e^{J_2(e^{-i\pi\hat{J}_1}\psi_2+e^{i\pi\hat{J}_1}\psi_2^\dagger)}\cdots,$$

as the fermionic operators ψ and $\psi^\dagger$ do not commute.

By the definition of J_k (2.14) and the properties of left vacuum state (2.19) and (2.20) we know that $\langle 0|\hat{J}_k = 0$ and hence use the definition of exponential function of operators, i.e.

$$e^{i\pi\hat{J}_{k-1}} = 1 + i\pi\hat{J}_{k-1} + \frac{(i\pi\hat{J}_{k-1})^2}{2!} + \cdots,$$

we have

$$G_t(\vec{J}) = \langle 0|\prod_k e^{J_k(\psi_k^\dagger+\psi_k)}e^{-\mathfrak{L}t}|0\rangle.$$

By defining $\psi_k^\dagger + \psi_k = \hat{S}_k$, we have

$$G_t(\vec{J}) = \langle 0|\prod_k e^{J_k\hat{S}_k}e^{-\mathfrak{L}t}|0\rangle.$$

Since

$$\hat{S}_k^2 = \left(\psi_k^\dagger + \psi_k\right)^2 = \psi_k^{\dagger 2} + \psi_k^2 + \left\{\psi_k^\dagger, \psi_k\right\} = 1,$$

we get

$$\mathrm{E}_t\left(s_{k_1}s_{k_2}\cdots s_{k_n}\right) = \prod_{j=1}^n \partial_{J_{x_j}} G_t(\vec{J})\mid_{\vec{J}=0} = \langle 0|\hat{S}_{x_1}\hat{S}_{x_2}\cdots\hat{S}_{x_n}e^{-\mathfrak{L}t}|0\rangle \tag{2.22}$$

for positions $x_n \geq x_{n-1} \geq \cdots \geq x_1$.

By defining $\hat{S}(t) = e^{\mathfrak{L}t}\hat{S}e^{-\mathfrak{L}t}$ we can rewrite (2.22) as

$$\mathrm{E}_t\left(s_{x_1}s_{x_2}\cdots s_{x_n}\right) = \langle 0|\hat{S}_{x_1}(t)\hat{S}_{x_2}(t)\cdots\hat{S}_{x_n}(t)|0\rangle.$$

By the anti-commutations (2.15) and (2.16) we can also rewrite (2.22) in terms of ψ operators:

$$\mathrm{E}_t\left(s_{x_1}s_{x_2}\cdots s_{x_n}\right) = \langle 0|\psi_{x_1}\psi_{x_2}\cdots\psi_{x_n}e^{-\mathfrak{L}t}|0\rangle.$$

or

$$\mathrm{E}_t\left(s_{x_1}s_{x_2}\cdots s_{x_n}\right) = \langle 0|\psi_{x_1}(t)\psi_{x_2}(t)\cdots\psi_{x_n}(t)|0\rangle. \tag{2.23}$$

where $\psi(t) = e^{\mathfrak{L}t}\psi e^{-\mathfrak{L}t}$, for $x_n > x_{n-1} > \cdots > x_1$.

2.1.3 Investigate the property of fermionic operators $\psi(t)$ and $\psi^\dagger(t)$

Since (2.23) gives the spin correlation we have to further investigate the property of the operators $\psi_k(t)$ and $\psi_k^\dagger(t)$. Firstly let us define an infinite dimensional vector of fermionic operators:

$$\vec{\phi} := (\cdots, \phi_{-2}, \phi_{-1}, \phi_1, \phi_2, \cdots) = \left(\cdots, \psi_2, \psi_1, \psi_1^\dagger, \psi_2^\dagger, \cdots\right)$$

and

$$\vec{\phi}\,(t) := e^{\mathfrak{L}t}\vec{\phi}e^{-\mathfrak{L}t}.$$

Lemma 3. *The fermionic operators consist of two parts: one part $\phi_k^+(t)$ annihilates the left vacuum state $\langle 0|$ and another $\phi_k^-(t)$ annihilates the right vacuum state $|0\rangle$.*

Proof As the Liouvillian operator $\mathfrak{L}$ is quadratic in terms of fermionic operators, it can be written in the bilinear form:

$$\mathfrak{L} = \vec{\phi}^T \underline{L} \vec{\phi} \tag{2.24}$$

where $\underline{L}$ is an infinite dimensional square matrix.

The commutation relations (2.15) and (2.16) can be combined as

$$\{\phi_k, \phi_l\} = \delta_{k+l} \tag{2.25}$$

and hence by direct calculation we can get

$$[\mathfrak{L}, \phi_k] = \sum_m \left(L_{m,-k} - L_{-k,m}\right)\phi_m. \tag{2.26}$$

By introducing a matrix $\underline{Ad_L}$ we can rewrite (2.26) as

$$\left(\underline{Ad_L}\right)_{k,m} = L_{m,-k} - L_{-k,m}. \tag{2.27}$$

By (2.27) and Hadamard's lemma, we have

$$\vec{\phi}(t) = e^{\mathfrak{L}t}\vec{\phi}e^{-\mathfrak{L}t} = \vec{\phi} + \frac{t}{1!}\left[\mathfrak{L}, \vec{\phi}\right] + \frac{t^2}{2!}\left[\mathfrak{L}, \left[\mathfrak{L}, \vec{\phi}\right]\right] + \cdots = e^{\underline{Ad_L}t}\vec{\phi}$$

and we can denote $\left(E(t)\right)_{i,j} = \left(e^{\underline{Ad_L}t}\right)_{i,j}$. Therefore, a fermionic operator $\phi_k(t)$ consists of two parts: one part $\phi_k^+(t)$ annihilates the left vacuum state $\langle 0|$ and another $\phi_k^-(t)$ annihilates the right vacuum state $|0\rangle$:

$$\phi_k(t) = \phi_k^+(t) + \phi_k^-(t)$$

where

$$\begin{aligned}\phi_k^+(t) &= \sum_{i=1}^{\infty} (E(t))_{k,i}\, \phi_i \\ \phi_k^-(t) &= \sum_{i=-1}^{-\infty} (E(t))_{k,i}\, \phi_i.\end{aligned}$$

□

Now that we know more about the property of ψ and $\psi^\dagger$ operators we can prove a useful lemma:

Lemma 4.

$$\left\{\psi_i^-(t), \psi_j(t)\right\} = E_t(s_i s_j)$$

Proof Firstly let us consider $\mathrm{E}(s_i s_j)$. From Lemma 3 we know that

$$\begin{aligned}\mathrm{E}_t(s_i s_j) &= \langle 0|\phi_i(t)\,\phi_j(t)\,|0\rangle \\ &= \sum_{q>0,p<0} (E(t))_{i,p}\,(E(t))_{i,q}\,\langle 0|\phi_p\phi_q|0\rangle.\end{aligned}$$

where $i, j < 0$. By (2.25) and (2.20) we have

$$\begin{aligned}\mathrm{E}_t(s_i s_j) &= \sum_{q>0,p<0} (E(t))_{i,p}\,(E(t))_{i,q}\,\langle 0|\delta_{p+q} - \phi_q\phi_p|0\rangle \\ &= \sum_{p<0} (E(t))_{i,p}\,(E(t))_{j,-p}\end{aligned}$$

By definition of $\phi_i^-(t)$ and $\phi_j(t)$ and direct calculation we have

$$\begin{aligned}\left\{\psi_i^-(t), \psi_j(t)\right\} &= \left\{\phi_i^-(t), \phi_j(t)\right\} \\ &= \sum_{p<0} (E(t))_{i,p}\,(E(t))_{j,-p} \\ &= \mathrm{E}_t(s_i s_j).\end{aligned}$$

□

2.1.4 Proof of the Pfaffian property of spin correlation

Having proved Lemma 3 we can use this result to prove the Pfaffian structure of spin correlation function $\mathrm{E}_t(s_{x_1}\cdots s_{x_n})$. The following theorem is very useful in the following chapters because we can use it to prove the Pfaffian structure of ARW/CRW.

Theorem 5. *The spin correlation $E_t\left(s_{x_1}\cdots s_{x_n}\right)$ is a Pfaffian of a matrix S of 2-point functions, i.e.*

$$E_t\left(s_{x_1}\cdots s_{x_n}\right) = Pf(S)\,,$$

where S is a skew-symmetric matrix

$$S_{i,j} = (-1)^{\chi(i<j)} E_t\left(s_i s_j\right),$$

where χ is an indicator function.

Proof Firstly we notice that if n is an odd number the spin correlation function $E_t\left(s_{i_1}\cdots s_{i_n}\right)$ is zero. This can be proved by observing $E_t\left(s\right) = 0$ and induction.

Let us denote $C_t\left(x_1,\cdots,x_n\right) = \mathrm{E}_t\left(s_{x_1}\cdots s_{x_n}\right)$. By Lemma 3 we know that $\psi_{x_1}(t) = \psi^-_{x_1}(t) + \psi^+_{x_1}(t)$ and $\langle 0|\psi^+_{x_1}(t) = 0$ and hence

$$C_t\left(x_1,\cdots,x_n\right) = \langle 0|\psi_{x_1}(t)\cdots\psi_{x_n}(t)|0\rangle = \langle 0|\psi^-_{x_1}(t)\cdots\psi_{x_n}(t)|0\rangle$$

where $\psi^-_{x_1}(t) = \sum_{i=1}^{\infty}\left(E(t)\right)_{x_1,i}\psi_i$. By lemma 4 we can obtain

$$\{\psi^-_{x_1}(t), \psi_{x_2}(t)\} = C_t\left(x_1,x_2\right).$$

Thus

$$\psi^-_{x_1}(t)\psi_{x_2}(t) = C_t\left(x_1,x_2\right) - \psi_{x_2}(t)\psi^-_{x_1}(t).$$

We keep permuting $\psi^-_{x_1}(t)$ to the right until it reaches the right vacuum state $|0\rangle$ and this gives

$$C_t\left(x_1,\cdots,x_n\right) = \sum_{k=2}^{n}(-1)^k(-1)^{\chi(x_1>x_k)}C_t\left(x_1,x_k\right)C_t\left(x_2,\cdots,x_{k-1},x_{k+1},\cdots,x_n\right)$$

which is the recursion expression of a Pfaffian. □

2.2 Annihilating random walk

Annihilating random walk (ARW) is a system of n particles performing random walk on a one-dimensional discrete lattice Z such that the probability of any particle walking in the positive direction in a small duration of time δt is $p\delta t$ while that in the negative direction is $(1-p)\,\delta t$. When two of the particles coincide on a site, they annihilate each other and disappear from the system. In this section our ARW is considered to be a symmetric one, i.e. $p = 1/2$ and the reaction time for the particles to merge together is taken to be infinitely fast.

In Glauber model, define a domain wall at position k to be $n_k = \frac{1-s_k s_{k+1}}{2}$. We can see that

$$n_k = \frac{1 - s_k s_{k+1}}{2} = \begin{cases} 1 & \text{if } s_k = -s_{k+1} \\ 0 & \text{if } s_k = s_{k+1} \end{cases}.$$

Therefore a domain wall can be regarded as a particle of ARW on a lattice isomorphic to Z since if two domain walls meet each other the spins would align and hence the two domain walls would annihilate each other.

Since for every spin s_i we have a corresponding operator $\hat{S}_i$, we can also define a corresponding operator for a particle in ARW n_k:

$$\hat{N}_k = \frac{1 - \hat{S}_k \hat{S}_{k+1}}{2}.$$

We can prove that this operator can give us the n-point correlation function of ARW. Firstly let us define what correlation function is for ARW.

Definition 3. *Particle density/n-point correlation function $\rho_n(x_1, \ldots, x_n; t)$ is the probability that all the positions x_k are occupied at time t. In other words,*

$$\rho_n(x_1, \ldots, x_n; t) = E_t\left(\prod_{i=1}^{n} \delta(x_n)\right)$$

where

$$\delta(x_i) = \begin{cases} 1, & \textit{if } x_i \textit{ is occupied} \\ 0, & \textit{otherwise.} \end{cases}$$

Now we can prove that from the $\hat{N}$ operator defined above we can get the n-point correlation function of ARW. In the case of ARW we denote our n-point correlation function as ρ_n^{ARW}.

Lemma 6.

$$\rho_n^{ARW}(x_1, \ldots, x_n; t) = E\left(\prod_{k=1}^{2n} n_{x_k}\right) = \langle 0|\prod_{k=1}^{2n} \hat{N}_{x_k} e^{-\mathfrak{L}t}|0\rangle$$

Proof It can be shown directly by expanding and colleciting all the terms.

$$
\begin{aligned}
\mathrm{E}\left(\prod_{k=1}^{2n} n_{x_k}\right) &= \frac{1}{2^{2n}}\mathrm{E}\left(\prod_{k=1}^{2n}\left(1-s_{x_k}s_{x_{k+1}}\right)\right) \\
&= \frac{1}{2^{2n}}\mathrm{E}\left\{1-\sum_{i=1}^{2n} s_{x_i}s_{x_i+1}+\sum_{\substack{i,k=1\\ i\neq k}}^{2n} s_{x_i}s_{x_i+1}s_{x_k}s_{x_k+1}+\cdots-\prod_{i=1}^{2n} s_{x_i}s_{x_i+1}\right\} \\
&= \frac{1}{2^{2n}}\{1-\sum_{i=1}^{2n}\langle 0|\hat{S}_{x_i}\hat{S}_{x_i+1}e^{-\mathfrak{L}t}|0\rangle+\sum_{\substack{i,k=1\\ i\neq k}}^{2n}\langle 0|\hat{S}_{s_i}\hat{S}_{s_i+1}\hat{S}_{s_k}\hat{S}_{s_k+1}e^{-\mathfrak{L}t}|0\rangle \\
&\quad +\cdots-\langle 0|\prod_{i=1}^{2n}\hat{S}_{x_i}\hat{S}_{x_i+1}e^{-\mathfrak{L}t}|0\rangle\} \\
&= \frac{1}{2^{2n}}\langle 0|\prod_{k=1}^{2n}\left(1-\hat{S}_{x_i}\hat{S}_{x_i+1}\right)e^{-\mathfrak{L}t}|0\rangle \\
&= \langle 0|\prod_{k=1}^{2n}\hat{N}_{x_k}e^{-\mathfrak{L}t}|0\rangle
\end{aligned}
$$

□

Recall that $\hat{S}_k = \psi_k + \psi_k^\dagger$, therefore the $\hat{S}$ operators can also be decomposed in the way $\hat{S} = \hat{S}^- + \hat{S}^+$ such that $\langle 0|\hat{S}^- = 0$ and $\hat{S}^+|0\rangle = 0$. Next we are going to prove a useful lemma between the $\hat{S}$ operators and the 2-point correlation function of spins.

Lemma 7.

$$\left\{\hat{S}_i^-, \hat{S}_j\right\} = \left\{\hat{S}_i^-(t), \hat{S}_j(t)\right\} = E_t(s_i s_j).$$

where $\hat{S}_i^-(t) = e^{\mathfrak{L}t}\hat{S}_i^- e^{-\mathfrak{L}t}$ *and* $\hat{S}_j(t) = e^{\mathfrak{L}t}\hat{S}_j e^{-\mathfrak{L}t}$.

Proof To prove the lemma we have to study the fermionic operators ψ and $\psi^\dagger$ and hence the $\hat{S}$ operator in the Fourier space.

By using Fourier transform we have

$$\psi_k = \oint_\epsilon \frac{d\lambda\lambda^{-k}}{2\pi i\lambda}\psi(\lambda)$$

where $\psi(\lambda) = \sum_{n\in\mathbb{Z}}\lambda^n\psi_n$. Define $\psi(\lambda, t) = e^{\mathfrak{L}t}\psi(\lambda)e^{-\mathfrak{L}t}$ and by Hadamard's lemma we have

$$\begin{pmatrix} \psi\left(\lambda,t\right) \\ \psi^{\dagger}\left(\lambda,t\right) \end{pmatrix} = \begin{pmatrix} A_t\left(\lambda\right) & B_t\left(\lambda\right) \\ 0 & C_t\left(\lambda\right) \end{pmatrix} \begin{pmatrix} \psi\left(\lambda\right) \\ \psi^{\dagger}\left(\lambda\right) \end{pmatrix}$$

where $A_t\left(\lambda,t\right)=e^{-t(1+\gamma(\lambda^{-1}+\lambda))}$, $B_t\left(\lambda,t\right)=\left(2\gamma\left(\lambda-\lambda^{-1}\right)\right)\frac{sinh\left(t\left(1+\gamma(\lambda^{-1}+\lambda)\right)\right)}{1+\gamma(\lambda^{-1}+\lambda)}$ and $C_t\left(\lambda,t\right)=e^{t(1+\gamma(\lambda^{-1}+\lambda))}$.

Therefore, we have

$$\begin{aligned} & \oint_{\epsilon}\frac{d\lambda\lambda^{-k}}{2\pi i\lambda}e^{-\mathfrak{L}t}\psi(\lambda,t)e^{\mathfrak{L}t} \\ = & \oint_{\epsilon}\frac{d\lambda\lambda^{-k}}{2\pi i\lambda}e^{-\mathfrak{L}t}\left[A_t(\lambda)\psi(\lambda)+B_t(\lambda)\psi^{\dagger}(\lambda)\right]e^{\mathfrak{L}t}. \end{aligned}$$

So we can obtain

$$\psi_k^{-}\left(t\right)=\oint_{\epsilon}\frac{d\lambda\lambda^{-k}}{2\pi i\lambda}A_t(\lambda)\psi(\lambda)$$

and

$$\psi_k^{+}\left(t\right)=\oint_{\epsilon}\frac{d\lambda\lambda^{-k}}{2\pi i\lambda}B_t(\lambda)\psi^{\dagger}(\lambda)$$

and notice that $\psi_k^{-}\left(t\right)\left|0\right\rangle=0$.

By a similar method we can obtain

$$\begin{aligned} \psi_k^{\dagger} &= \oint_{\epsilon}\frac{d\lambda\lambda^{-k}}{2\pi i\lambda}\psi^{\dagger}(\lambda) \\ &= \oint_{\epsilon}\frac{d\lambda\lambda^{-k}}{2\pi i\lambda}e^{-\mathfrak{L}t}\psi^{\dagger}(\lambda,t)e^{\mathfrak{L}t} \\ &= \oint_{\epsilon}\frac{d\lambda\lambda^{-k}}{2\pi i\lambda}e^{-\mathfrak{L}t}C_t(\lambda)\psi^{\dagger}(\lambda)e^{\mathfrak{L}t}. \end{aligned}$$

Hence

$$\psi_k^{\dagger}\left(t\right)=\oint_{\epsilon}\frac{d\lambda\lambda^{-k}}{2\pi i\lambda}C_t(\lambda)\psi^{\dagger}(\lambda)$$

and

$$\left\langle 0\right|\psi_k^{\dagger}\left(t\right)=0.$$

Hence we know that

$$\hat{S}_k^{-}\left(t\right)=\psi_k^{-}\left(t\right).$$

Now we can preceed to the following:

$$\begin{aligned}\left\{\hat{S}_i^-(t), \hat{S}_j(t)\right\} &= \left\{\psi_i^-(t), \psi_j(t) + \psi_j^\dagger(t)\right\} \\ &= \left\{\psi_i^-(t), \psi_j(t)\right\} + \left\{\psi_i^-(t), \psi_j^\dagger(t)\right\}.\end{aligned}$$

By Lemma 4 the first term on the right hand side $\left\{\psi_i^-(t), \psi_j(t)\right\}$ is known to be $\mathrm{E}_t(s_i s_j)$ for $j > i$. For $j = i$, $\left\{\psi_i^-, \psi_j\right\}$ is equal to zero. So now we have to show that $\left\{\psi_i^-(t), \psi_j^\dagger(t)\right\} = 0$ for $j > i$.

$$\begin{aligned}\left\{\psi_i^-(t), \psi_j^\dagger(t)\right\} &= \oint_\epsilon \oint_\epsilon \left(\frac{d\lambda\lambda^{-i}}{2\pi i\lambda}\right)\left(\frac{d\mu\mu^{-j}}{2\pi i\mu}\right) A_t(\lambda)C_t(\mu)\left\{\psi(\lambda), \psi^\dagger(\mu)\right\} \\ &= \oint_\epsilon \oint_\epsilon \left(\frac{d\lambda\lambda^{-i}}{2\pi i\lambda}\right)\left(\frac{d\mu\mu^{-j}}{2\pi i\mu}\right) A_t(\lambda)C_t(\mu)\delta(\lambda\mu) \\ &= \oint_\epsilon \left(\frac{d\lambda\lambda^{-i}}{2\pi i\lambda}\right)\lambda^j A_t(\lambda)C_t(\lambda^{-1}).\end{aligned}$$

Now let us investigate the exact form of the function $A_t(\lambda)$ and $C_t(\lambda^{-1})$:

$$\begin{aligned}A_t(\lambda)C_t(\lambda^{-1}) &= \exp\left\{-t\left[1+\gamma\left(\lambda+\lambda^{-1}\right)\right]\right\}\exp\left\{t\left[1+\gamma\left(\left(\lambda^{-1}\right)+\left(\lambda^{-1}\right)^{-1}\right)\right]\right\} \\ &= 1.\end{aligned}$$

So $\left\{\psi_i^{(-)}, \psi_j^\dagger\right\} = \oint_\epsilon \frac{d\lambda\lambda^{j-i}}{2\pi i\lambda} = 0$ for $j > i$.

For $j = i$, it is obvious that $\left\{\psi_i^{(-)}, \psi_i^\dagger\right\} = \oint_\epsilon \frac{d\lambda}{2\pi i\lambda} = 1$. So for $i = j$, the anticommutator $\left\{\hat{S}_i^-(t), \hat{S}_i(t)\right\} = 1 = \mathrm{E}_t\left(s_i^2\right)$, which coincide the definition of a spin as a random variable.

Therefore, we can conclude that for $j \geq i$,

$$\left\{\hat{S}_i^-(t), \hat{S}_j(t)\right\} = \mathrm{E}_t(s_i s_j).$$

and hence

$$\left\{\hat{S}_i^-, \hat{S}_j\right\} = e^{-\mathfrak{L}t}\left\{\hat{S}_i^-(t), \hat{S}_j(t)\right\}e^{\mathfrak{L}t} = \mathrm{E}_t(s_i s_j).$$

□

Remark By the Fourier transform used in the above proof, we can obtain the integral form of 2-point function.

$$\begin{aligned}
\mathrm{E}_t\left(s_i s_j\right) &= \left\{\psi_i^-(t), \psi_j(t)\right\} \\
&= \oint_\epsilon \oint_\epsilon \left(\frac{d\lambda \lambda^{-i}}{2\pi i \lambda}\right)\left(\frac{d\mu \mu^{-j}}{2\pi i \mu}\right) A_t(\lambda) B_t(\mu) \delta(\lambda\mu) \\
&= \oint_\epsilon \frac{d\mu}{2\pi i \mu} \mu^{j-i} D(\mu)
\end{aligned}$$

where

$$D(\mu) = \gamma\left(\mu - \mu^{-1}\right) \frac{1 - e^{-2t\left(1+\gamma\left(\mu^{-1}+\mu\right)\right)}}{1 + \gamma\left(\mu^{-1}+\mu\right)}$$

and $j - i > 0$.

2-point function at zero temperature The integral is ill-defined at zero temperature $\gamma = \frac{-1}{2}$ because of the double-pole at $\mu = 1$. However, we can let $\gamma = \frac{-1}{2} + \epsilon$, where $\epsilon > 0$, and then let ϵ tend to zero. Then

$$D(\mu) = \frac{\mu^2 - 1}{(\mu - 1)^2 - \epsilon}\left(1 - e^{-2t} e^{t\left(\mu^{-1}+\mu\right)}\right).$$

Therefore the 2-point function at zero temperature $\gamma = \frac{-1}{2}$ becomes

$$\mathrm{E}_t\left(s_i s_j\right) = 1 - e^{-2t} \oint \frac{d\mu}{2\pi i} \frac{\mu^{-(j-i)} - \mu^{(j-i)}}{1 - \mu} e^{t\left(\mu^{-1}+\mu\right)} \tag{2.28}$$

where we have assumed $j - i > 0$.

Having Lemma 6 and Lemma 7 at our disposal, we can prove the n-point correlation function of ARW has an interesting property in the special case that all the particle are next to each other and n is an odd number:

Theorem 8. *The $2m+1$-point correlation function $\rho_n^{ARW}\left(x_1, \ldots, x_{2m+1}; t\right)$ of ARW can be expressed as a Pfaffian of a $(2m+2) \times (2m+2)$ matrix, i.e.*

$$\rho_n^{ARW}\left(x_1, \ldots, x_{2m+1}; t\right) = \frac{1}{2^m} Pf_{1 \le k, l \le 2m+2}\left((-1)^{\chi(k>l)} f_{k,l}\right),$$

where $f_{k,l} = \frac{1 - E_t\left(s_{x_k} s_{x_l}\right)}{2}$ and $x_k = x_{k-1} + 1$.

Proof To simplify the notations and for the convenience of the proof, let

$$s_k = s_{x_k} \text{ and } n_{k,l} = \frac{1 - s_k s_l}{2}$$

and

$$C\left(k,l\right)=\mathrm{E}_{t}\left(s_{k}s_{l}\right).$$

$$\begin{aligned}
& \rho_{n}^{ARW}\left(x_{1},\ldots,x_{2m+1};t\right) \\
= \;& \mathrm{E}_{t}\left(\prod_{k=1}^{2m+1}n_{k,k+1}\right) \\
= \;& \langle 0|\prod_{k=1}^{2m+1}\hat{N}_{k}e^{-\mathfrak{L}t}|0\rangle \\
= \;& \frac{1}{2}\langle 0|\prod_{k=2}^{2m+1}\hat{N}_{k}e^{-\mathfrak{L}t}|0\rangle-\frac{1}{2}\langle 0|\hat{S}_{1}^{-}\hat{S}_{2}\prod_{k=2}^{2m+1}\hat{N}_{k}e^{-\mathfrak{L}t}|0\rangle \\
= \;& \frac{1}{2}\mathrm{E}_{t}\left(\prod_{k=2}^{2m+1}n_{k}\right)-\frac{1}{2}\langle 0|\left(\left\{\hat{S}_{1}^{-},\hat{S}_{2}\right\}-\hat{S}_{2}\hat{S}_{1}^{-}\right)\prod_{k=2}^{2m+1}\hat{N}_{k}e^{-\mathfrak{L}t}|0\rangle \\
= \;& \frac{\left(1-C\left(1,2\right)\right)}{2}\mathrm{E}_{t}\left(\prod_{k=2}^{2m+1}n_{k}\right)+\frac{1}{2}\langle 0|\hat{S}_{2}\left[\hat{S}_{1}^{-},\prod_{k=2}^{2m+1}\hat{N}_{k}\right]e^{-\mathfrak{L}t}|0\rangle \\
= \;& f_{1,2}\mathrm{E}_{t}\left(\prod_{k=2}^{2m+1}n_{k}\right)+\frac{1}{2}\sum_{p=2}^{2m+1}\langle 0|\hat{S}_{2}\prod_{k=2}^{p-1}\hat{N}_{k}\left[\hat{S}_{1}^{-},\hat{N}_{p}\right]\prod_{q=p+1}^{2m+1}\hat{N}_{q}e^{-\mathfrak{L}t}|0\rangle \\
= \;& f_{1,2}\mathrm{E}_{t}\left(\prod_{k=2}^{2m+1}n_{k}\right) \\
& +\frac{1}{2}\sum_{p=2}^{2m+1}\langle 0|\hat{S}_{2}\prod_{k=2}^{p-1}\hat{N}_{k}\frac{\left(C\left(1,p+1\right)\hat{S}_{p}-C\left(1,p\right)\hat{S}_{p+1}\right)}{2}\prod_{q=p+1}^{2m+1}\hat{N}_{q}e^{-\mathfrak{L}t}|0\rangle \\
= \;& f_{1,2}\mathrm{E}_{t}\left(\prod_{k=2}^{2m+1}n_{k,k+1}\right) \\
& +\frac{1}{2}\sum_{p=2}^{2m+1}\mathrm{E}_{t}\left(s_{2}:\prod_{k=2}^{p-1}n_{k,k+1}\frac{\left(C\left(1,p+1\right)s_{p}-C\left(1,p\right)s_{p+1}\right)}{2}\prod_{q=p+1}^{2m+1}n_{q,q+1}\right)
\end{aligned}$$

$$
\begin{aligned}
= \quad & f_{1,2}\mathrm{E}_t\left(\prod_{k=2}^{2m+1} n_{k,k+1}\right) \\
& +\frac{1}{2}\sum_{p=2}^{2m+1}\mathrm{E}_t\left(s_2\prod_{k=2}^{p-1} n_{k,k+1}\frac{(s_p - s_{p+1})}{2}\prod_{q=p+1}^{2m+1} n_{q,q+1}\right) \\
& -\frac{1}{2}\sum_{p=2}^{2m+1}\mathrm{E}_t\left(s_2\prod_{k=2}^{p-1} n_{k,k+1}\left(f_{1,p+1}s_p - f_{1,p}s_{p+1}\right)\prod_{q=p+1}^{2m+1} n_{q,q+1}\right)
\end{aligned}
$$

The second term, of which is order zero in $f_{i,j}$, is zero. To show this we have to notice two facts. Firstly,

$$
s_p - s_{p+1} = s_p\left(1 - s_p s_{p+1}\right) = 2s_p n_{p,p+1}.
$$

And secondly,

$$
s_2\prod_{k=2}^{p-1} n_{k,k+1}s_p = (-1)^p\prod_{k=2}^{p-1} n_{k,k+1}
$$

since the product is non-zero only when all n_k are 1 and thus $s_2 = -s_3 = s_4 = \cdots = (-1)^p s_p$. Using these two facts we have

$$
\begin{aligned}
& \frac{1}{2^2}\sum_{p=2}^{2m+1}\mathrm{E}_t\left(s_2\prod_{k=2}^{p-1} n_{k,k+1}\left(s_p - s_{p+1}\right)\prod_{q=p+1}^{2m+1} n_{q,q+1}\right) \\
= \quad & \frac{1}{2}\sum_{p=2}^{2m+1}\mathrm{E}_t\left(s_2\prod_{k=2}^{p-1} n_{k,k+1}s_p n_{p.p+1}\prod_{q=p+1}^{2m+1} n_{q,q+1}\right) \\
= \quad & \left(\sum_{p=2}^{2m+1}(-1)^p\right)\cdot\mathrm{E}_t\left(\prod_{k=2}^{2m+1} n_{k,k+1}\right) \\
= \quad & 0
\end{aligned}
$$

since $\left(\sum_{p=2}^{2m+1}(-1)^p\right) = 0$ as there are even number of terms in the summation.

Therefore,

$$
\begin{aligned}
& \mathrm{E}_t\left(\prod_{k=1}^{2m+1} n_{k,k+1}\right) \\
= \quad & f_{1,2}\mathrm{E}_t\left(\prod_{k=2}^{2m+1} n_{k,k+1}\right) \\
& -\frac{1}{2}\sum_{p=2}^{2m+1} \mathrm{E}_t\left(s_2 \prod_{k=2}^{p-1} n_{k,k+1}\left(f_{1,p+1}s_p - f_{1,p}s_{p+1}\right)\prod_{q=p+1}^{2m+1} n_{q,q+1}\right) \\
= \quad & f_{1,2}\mathrm{E}_t\left(\prod_{k=2}^{2m+1} n_{k,k+1}\right) - \frac{1}{2}\mathrm{E}_t\left(s_2\left(f_{1,3}s_2 - f_{1,2}s_3\right)\prod_{q=3}^{2m+1} n_{q,q+1}\right) \\
& -\frac{1}{2}\mathrm{E}_t\left(s_2 : \prod_{k=2}^{2m} n_{k,k+1}\left(f_{1,2m+2}s_{2m+1} - f_{1,2m+1}s_{2m+2}\right)\right) \\
& -\frac{1}{2}\sum_{p=3}^{2m}\mathrm{E}_t\left(s_2 \prod_{k=2}^{p-1} n_{k,k+1}\left(f_{1,p+1}s_p - f_{1,p}s_{p+1}\right)\prod_{q=p+1}^{2m+1} n_{q,q+1}\right) \\
= \quad & \frac{1}{2}f_{1,2}\mathrm{E}_t\left(\prod_{k=3}^{2m+1} n_{k,k+1}\right) - \frac{1}{2}f_{1,3}\mathrm{E}_t\left(\prod_{q=3}^{2m+1} n_{q,q+1}\right) \\
& -\frac{1}{2}\mathrm{E}_t\left(s_2 \prod_{k=2}^{2m} n_{k,k+1}\left(f_{1,2m+2}s_{2m+1} - f_{1,2m+1}s_{2m+2}\right)\right) \\
& -\frac{1}{2}\sum_{p=4}^{2m+1} f_{1,p}\mathrm{E}_t\left(s_2 \prod_{k=2}^{p-2} n_{k,k+1}s_{p-1} \prod_{q=p}^{2m+1} n_{q,q+1}\right) \\
& +\frac{1}{2}\sum_{p=3}^{2m} f_{1,p}\mathrm{E}_t\left(s_2 \prod_{k=2}^{p-1} n_{k,k+1}s_{p+1} \prod_{q=p+1}^{2m+1} n_{q,q+1}\right)
\end{aligned}
$$

$$\begin{aligned}
= \;& \frac{1}{2} f_{1,2} \mathrm{E}_t \left(\prod_{k=3}^{2m+1} n_{k,k+1} \right) - \frac{1}{2} f_{1,3} \mathrm{E}_t \left(\prod_{q=3}^{2m+1} n_{q,q+1} \right) \\
& - \frac{1}{2} \mathrm{E}_t \left(s_2 \prod_{k=2}^{2m} n_{k,k+1} \left(f_{1,2m+2} s_{2m+1} - f_{1,2m+1} s_{2m+2} \right) \right) \\
& - \frac{1}{2} f_{1,2m+1} \mathrm{E}_t \left(s_2 \prod_{k=2}^{2m-1} n_{k,k+1} s_{2m} n_{2m+1,2m+2} \right) \\
& + \frac{1}{2} f_{1,3} \mathrm{E}_t \left(s_2 n_{2,3} s_4 \prod_{q=4}^{2m+1} n_{q,q+1} \right) \\
& - \frac{1}{2} \sum_{p=4}^{2m} f_{1,p} \mathrm{E}_t \left(s_2 \prod_{k=2}^{p-2} n_{k,k+1} \left(s_{p-1} n_{p,p+1} - n_{p-1,p} s_{p+1} \right) \prod_{q=p+1}^{2m+1} n_{q,q+1} \right)
\end{aligned}$$

There are six terms in the expression above and we can simplify them one by one. Firstly, we can combine the second term and the fifth term as follows:

$$-\frac{1}{2} f_{1,3} \mathrm{E}_t \left(\prod_{q=3}^{2m+1} n_{q,q+1} \right) + \frac{1}{2} f_{1,3} \mathrm{E}_t \left(s_2 n_{2,3} s_4 \prod_{q=4}^{2m+1} n_{q,q+1} \right) = -\frac{1}{2} f_{1,3} \mathrm{E}_t \left(n_{2,4} \prod_{q=4}^{2m+1} n_{q,q+1} \right).$$

Secondly, we can combine the third and the fourth term as follows:

$$\begin{aligned}
& - \frac{1}{2} \mathrm{E}_t \left(s_2 \prod_{k=2}^{2m} n_{k,k+1} \left(f_{1,2m+2} s_{2m+1} - f_{1,2m+1} s_{2m+2} \right) \right) \\
& - \frac{1}{2} f_{1,2m+1} \mathrm{E}_t \left(s_2 \prod_{k=2}^{2m-1} n_{k,k+1} s_{2m} n_{2m+1,2m+2} \right) \\
= \;& \frac{1}{2} f_{1,2m+2} \mathrm{E}_t \left(\prod_{k=2}^{2m} n_{k,k+1} \right) - \frac{1}{2} f_{1,2m+1} \mathrm{E}_t \left(\prod_{k=2}^{2m-1} n_{k,k+1} n_{2m,2m+2} \right)
\end{aligned}$$

Thirdly, the sixth term can be changed as follows:

$$\begin{aligned}
& -\frac{1}{2}\sum_{p=4}^{2m} f_{1,p}\mathrm{E}_t\left(s_2\prod_{k=2}^{p-2} n_{k,k+1}\left(s_{p-1}n_{p,p+1}-n_{p-1,p}s_{p+1}\right)\prod_{q=p+1}^{2m+1} n_{q,q+1}\right) \\
= & -\frac{1}{2}\sum_{p=4}^{2m} f_{1,p}\mathrm{E}_t\left(s_2\prod_{k=2}^{p-2} n_{k,k+1}s_{p-1}n_{p-1,p+1}\prod_{q=p+1}^{2m+1} n_{q,q+1}\right) \\
= & \ \frac{1}{2}\sum_{p=4}^{2m}\left(-1\right)^p f_{1,p}\mathrm{E}_t\left(\prod_{k=2}^{p-2} n_{k,k+1}n_{p-1,p+1}\prod_{q=p+1}^{2m+1} n_{q,q+1}\right).
\end{aligned}$$

Finally, summing up all the contributions we get

$$\mathrm{E}_t\left(\prod_{k=1}^{2m+1} n_{k,k+1}\right)=\frac{1}{2}\sum_{p=2}^{2m+2}\left(-1\right)^p f_{1,p}\mathrm{E}_t\left(\prod_{k=2}^{p-2} n_{k,k+1}n_{p-1,p+1}\prod_{q=p+1}^{2m+1} n_{q,q+1}\right). \tag{2.29}$$

It might not seem obvious that equation (2.29) shows the correlation function is a Pfaffian but if we introduce the notation

$$A^{(2m+2)}\left(x_1,x_2,\ldots,x_{2m+2}\right)=\mathrm{E}_t\left(\prod_{k=1}^{2m+1} n_{k,k+1}\right)$$

then we can change (2.29) to the following form:

$$A^{(2m+2)}\left(x_1,x_2,\ldots,x_{2m+2}\right)=\frac{1}{2}\sum_{k=2}^{2m+2}\left(-1\right)^k A^{(2m+1)}\left(x_2,x_3,\ldots,x_{k-1},x_{k+1},x_{k+2},\ldots,x_{2m+2}\right)$$

which coincides with the standard recursion relation between Pfaffians except the extra factor $\frac{1}{2}$. Therefore this proves the statement. □

However, this property will not be used in the proof that ARW is a Pfaffian point process in the next chapter but it gives the hint that the general n-point correlation function might have a Pfaffian structure. We will see that in the following chapter.

Chapter 3

Coalescing Random Walk and Glauber Model

In this chapter we are going to investigate the relationship between CRW and Glauber model by showing that an object called empty interval probability is identical to the spin correlation in Glauber model. The particle correlation function for CRW can then be derived from this object and will be shown in the next chapter.

Coalescing random walk (CRW) is a system of n particles performing independent random walk on a one-dimensional discrete lattice $Z + 0.5$ such that the probability of any particle walking in the positive direction in a small duration of time δt is $p\delta t$ while that in the negative direction is $(1-p)\,\delta t$. When two of the particles coincide on a site, they merge into one particle and continue to perform random walk. In this thesis our CRW is considered to be a symmetric one, i.e. $p = 1/2$ and the reaction rate for the particles to merge together is taken to be infinitely fast.

Let us define empty interval probability:

Definition 4. *Let Ω_{x_i,y_i} denotes the event that the positions $\{x^* \in Z + 0.5 : x_i < x^* < y_i; x_i, y_i \in Z\}$ are empty. The empty interval probability, denote by $P_t\left[\Omega_{x_1,y_1} \cap \cdots \cap \Omega_{x_n,y_n}\right]$, is the probability that the positions $\{x^* \in Z + 0.5 : x_i < x^* < y_i; x_i, y_i \in Z\}$ for all $i = 1, \ldots, n$ are empty at time t.*

For example, $P_t\left[\Omega_{1,3}\right]$ is the probability that the positions $x = \{1.5, 2.5\}$ are not occupied by particles at time t. Also it is true that $P_t\left[\Omega_{x,x}\right] = 1$. The name "empty interval" originates from the continuous case. We can define our "empty interval" in our discrete case as

$$(x, y) := \{x^* \in \mathrm{Z} + 0.5 : x_i < x^* < y_i; x_i, y_i \in \mathrm{Z}\}. \tag{3.1}$$

In the following section the word "interval" is understood as the empty interval defined in (3.1).

Having the definition of empty interval probability, we want to prove the following theorem:

Theorem 9. *The probability $P_t\left[\Omega_{x_1,y_1} \cap \cdots \cap \Omega_{x_n,y_n}\right]$ and the spin correlation function $E\left(s_{x_1}s_{y_1} \cdots s_{x_n}s_{y_n}\right)$ both satisfy the heat equation and the same set of boundary conditions and hence are identical equations by the uniqueness theorem of the heat equation, i.e.*

$$P_t\left(\Omega_{x_1,y_1} \cap \cdots \cap \Omega_{x_n,y_n}\right) = E_t\left(s_{x_1}s_{y_n} \cdots s_{x_n}s_{y_n}\right)$$

if given the same initial condition

$$P_0\left(\Omega_{x_1,y_1} \cap \cdots \cap \Omega_{x_n,y_n}\right) = E_0\left(s_{x_1}s_{y_n} \cdots s_{x_n}s_{y_n}\right)$$

where $x_1 < y_1 < x_2 < y_2 < \cdots < x_n < y_n$.

For example, if all the intervals (x_i, y_i) for $i = 1, \ldots, n$ are empty and $s_{x_i} = s_{y_i} = 1$ for all $i = 1, \ldots, n$ then the initial conditions are the same, i.e.

$$P_0\left(\Omega_{x_1,y_1} \cap \cdots \cap \Omega_{x_n,y_n}\right) = \mathrm{E}_0\left(s_{x_1}s_{y_n} \cdots s_{x_n}s_{y_n}\right) = 1.$$

Proof

To show this, we have to prove that both equations satisfy the heat equation and the same set of boundary conditions and hence are identical equations by the uniqueness of the heat equation.

3.1 Kinetic equation for empty interval probability

We want to derive the kinetic equation for the empty intervals. For a small time duration δt, the only contribution to the change of the empty interval probability is from the particles hopping in and out of the intervals at the edges.

Suppose now we have no particle in (x, y) but there is a particle at $y + 0.5$ at time t. We can express the probability of this event by the difference of two empty interval probabilities:

$$P_t\left(\Omega_{x,y} \setminus \Omega_{x,y+1}\right) = P_t\left(\Omega_{x,y}\right) - P_t\left(\Omega_{x,y+1}\right)$$

since $\Omega_{x,y} \supset \Omega_{x,y+1}$. Similarly if there is no particle in (x, y) but there is a particle at $x - 0.5$ at time t. We can express the probability of this event by:

$$P_t\left(\Omega_{x,y} \setminus \Omega_{x-1,y}\right) = P_t\left(\Omega_{x,y}\right) - P_t\left(\Omega_{x-1,y}\right).$$

Consider an interval (x_i, y_i) in n disjoint intervals $(x_1, y_1) \ldots (x_n, y_n)$. Let D be the hopping rate per unit time. The increase in the probability $P_t\left(\Omega_{x_1,y_1} \cap \cdots \cap \Omega_{x_n,y_n}\right)$ in a small duration of time δt is due to a particle hopping out of the interval at the left boundary or of the right boundary, which are

$$\left[P_t\left(\Omega_{x_1,y_1} \cup \cdots \cup \Omega_{x_i+1,y_i} \cup \cdots \cup \Omega_{x_n,y_n}\right) - P_t\left(\Omega_{x_1,y_1} \cup \cdots \cup \Omega_{x_i,y_i} \cup \cdots \cup \Omega_{x_n,y_n}\right)\right] D\left(\delta t\right) \tag{3.2}$$

and

$$\left[P_t\left(\Omega_{x_1,y_1} \cup \cdots \cup \Omega_{x_i,y_i-1} \cup \cdots \cup \Omega_{x_n,y_n}\right) - P_t\left(\Omega_{x_1,y_1} \cup \cdots \cup \Omega_{x_i,y_i} \cup \cdots \cup \Omega_{x_n,y_n}\right)\right] D\left(\delta t\right) \tag{3.3}$$

respectively. In a similar fashion, the decrease in the probability $P_t\left(\Omega_{x_1,y_1} \cap \cdots \cap \Omega_{x_n,y_n}\right)$ in δt due to a particles hopping in the interval at the left boundary or the right boundary are

$$-\left[P_t\left(\Omega_{x_1,y_1} \cup \cdots \cup \Omega_{x_i,y_i} \cup \cdots \cup \Omega_{x_n,y_n}\right) - P_t\left(\Omega_{x_1,y_1} \cup \cdots \cup \Omega_{x_i-1,y_i} \cup \cdots \cup \Omega_{x_n,y_n}\right)\right] D\left(\delta t\right) \tag{3.4}$$

and

$$-\left[P_t\left(\Omega_{x_1,y_1} \cup \cdots \cup \Omega_{x,y} \cup \cdots \cup \Omega_{x_n,y_n}\right) - P_t\left(\Omega_{x_1,y_1} \cup \cdots \cup \Omega_{x_i,y_i+1} \cup \cdots \cup \Omega_{x_n,y_n}\right)\right] D\left(\delta t\right) \tag{3.5}$$

respectively.

For brevity we can denote

$$P_t\left(\Omega_{x_1,y_1} \cap \cdots \cap \Omega_{x_n,y_n}\right)$$

by

$$P_t\left(\underline{x}, \underline{y}\right).$$

Also we can denote the forward discrete derivative $\partial^+_{x_i}$ and backward discrete derivative $\partial_{x_i}^-$ by

$$\partial_{x_i}^+ P_t\left(\underline{x},\underline{y}\right) = P_t\left(\cdots \cup \Omega_{x_i+1,y_i} \cup \dots\right) - P_t\left(\cdots \cup \Omega_{x_i,y_i} \cup \dots\right)$$

and

$$\partial_{x_i}^- P_t\left(\underline{x},\underline{y}\right) = P_t\left(\cdots \cup \Omega_{x_i,y_i} \cup \dots\right) - P_t\left(\cdots \cup \Omega_{x_{i-1},y_i} \cup \dots\right).$$

Summing up the contribution to the change in the probability $P_t\left(\underline{x},\underline{y}\right)$ in δt from the interval (x_i, y_i) in equations (3.2), (3.3), (3.4) and (3.5) we get

$$\left[\left(\partial_{x_i}^-\partial_{x_i}^+ + \partial_{y_i}^-\partial_{y_i}^+\right) P_t\left(\underline{x},\underline{y}\right)\right] D\left(\delta t\right). \tag{3.6}$$

In conclusion, if we collect the contributions from all the intervals (x_i, y_i) for $n = 1, \dots, n$ we have

$$\begin{aligned} & P_{t+\delta t}\left(\underline{x},\underline{y}\right) - P_t\left(\underline{x},\underline{y}\right) \\ = \quad & D\left(\delta t\right)\sum_{i=1}^{n}\left[\left(\partial_{x_i}^-\partial_{x_i}^+ + \partial_{y_i}^-\partial_{y_i}^+\right) P_t\left(\underline{x},\underline{y}\right)\right] \end{aligned}$$

Equation (3.6) is the discrete Laplacian with respect to variables x_i and y_i of the probability)$P_t\left(\underline{x},\underline{y}\right)$ times $D\left(\delta t\right)$. Therefore, by considering the total contribution from all the intervals we can get

$$\partial_t P_t\left(\underline{x},\underline{y}\right) = D\Delta P_t\left(\underline{x},\underline{y}\right) \;\; \forall t \geq 0 \tag{3.7}$$

for $x_1 < y_1 < x_2 < y_2 < \cdots < x_n < y_n$ and Δ stands for the discrete Laplacian with respect to all the boundary variables x_i and y_i, that is

$$\Delta = \sum_{i=1}^{n}\left(\partial_{x_i}^-\partial_{x_i}^+ + \partial_{y_i}^-\partial_{y_i}^+\right). \tag{3.8}$$

Now let us investigate the boundary conditions of the differential equation. Since $P_t\left[\Omega_{x,x}\right] = 1$, we have

$$P_t\left(\underline{x},\underline{y}\right) = P_t\left(\underline{x'},\underline{y'}\right) \;\text{ if } x_i = y_i \tag{3.9}$$

where $\underline{x'} = \underline{x} \setminus \{x_i\}$ and $\underline{y'} = \underline{y} \setminus \{y_i\}$.

For $y_i = x_{i+1}$, since $\{x^* \in \mathrm{Z} + 0.5 : x_i < x^* < y_i\} \cup \{x^* \in \mathrm{Z} + 0.5 : x_{i+1} < x^* <$

$y_{i+1}\} = \{x^* \in \mathrm{Z} + 0.5 : x_i < x^* < y_{i+1}\}$, we have

$$P_t\left(\underline{x}, \underline{y}\right) = P_t\left(\underline{\tilde{x}}, \underline{\tilde{y}}\right) \text{ if } y_i = x_{i+1} \tag{3.10}$$

where $\underline{\tilde{x}} = \underline{x} \setminus \{x_{i+1}\}$ and $\underline{\tilde{y}} = \underline{y} \setminus \{y_i\}$.

At this point we have obtained the kinetic equation for empty interval and the boundary conditions. We are going to investigate these equations for the equation of spin correlation.

3.2 Kinetic equation for the spin correlation

Let $\vec{s}$ be a spin configuration and $D\left[1 - \frac{s_x(s_{x-1}+s_{x+1})}{2}\right]$ be the flipping rate of the spin s_x at position x . In a short duration of time δt, the change of the probability of $\vec{s}$ at time t can only be due to a flip of a single spin (otherwise, there would be δt^2 contribution which can be ignored).

Therefore,

$$\begin{aligned} & P_{t+\delta t}\left(\vec{s}\right) - P_t\left(\vec{s}\right) \\ = & -D\left(\delta t\right)\sum_{x\in \mathrm{Z}}\left[1 - \frac{s_x\left(s_{x-1}+s_{x+1}\right)}{2}\right]P_t\left(\vec{s}\right) + D\left(\delta t\right)\sum_{x\in \mathrm{Z}}\left[1 + \frac{s_x\left(s_{x-1}+s_{x+1}\right)}{2}\right]P_t\left(\vec{\sigma_x}\right) + O\left(\delta t^2\right) \end{aligned}$$

where $\vec{\sigma_x}$ is the configuration of spins differs from $\vec{s}$ only at x. Rewriting the above equation we have

$$\partial_t P_t\left(\vec{s}\right) = -D\sum_{x\in \mathrm{Z}}\left[1 - \frac{s_x\left(s_{x-1}+s_{x+1}\right)}{2}\right]P_t\left(\vec{s}\right) + D\sum_{x\in \mathrm{Z}}\left[1 + \frac{s_x\left(s_{x-1}+s_{x+1}\right)}{2}\right]P_t\left(\vec{\sigma_x}\right). \tag{3.11}$$

Now consider $2n$ spins at the positions $k_1 < k_2 < \cdots < k_{2n}$. By using (3.11) we can write the change of the spin correlation with respect to time as

$$\begin{aligned} & \partial_t \mathrm{E}_t\left(s_{k_1}s_{k_2}\dots s_{k_{2n}}\right) \\ = & \sum_{\vec{s}}\partial_t P_t\left(\vec{s}\right)\left(s_{k_1}s_{k_2}\dots s_{k_{2n}}\right) \\ = & -\sum_{\vec{s}}\sum_{x\in \mathrm{Z}}D\left[1 - \frac{s_x\left(s_{x-1}+s_{x+1}\right)}{2}\right]P_t\left(\vec{s}\right)\left(s_{k_1}s_{k_2}\dots s_{k_{2n}}\right) \\ & +\sum_{\vec{s}}\sum_{x\in \mathrm{Z}}D\left[1 + \frac{s_x\left(s_{x-1}+s_{x+1}\right)}{2}\right]P_t\left(\vec{\sigma_x}\right)\left(s_{k_1}s_{k_2}\dots s_{k_{2n}}\right). \end{aligned}$$

If $x \notin \{k_1, k_2, \ldots, k_{2n}\}$,

$$D \sum_{\vec{s}} P_t(\vec{s})(s_{k_1} s_{k_2} \ldots s_{k_{2n}}) = D \sum_{\vec{s}} P_t(\vec{\sigma_x})(s_{k_1} s_{k_2} \ldots s_{k_{2n}}).$$

Otherwise, if $x \in \{k_1, k_2, \ldots, k_{2n}\}$,

$$-D \sum_{\vec{s}} P_t(\vec{s})(s_{k_1} s_{k_2} \ldots s_{k_{2n}}) = D \sum_{\vec{s}} P_t(\vec{\sigma_x})(s_{k_1} s_{k_2} \ldots s_{k_{2n}}).$$

Therefore,

$$-D \sum_{x \in \mathrm{Z}} \sum_{\vec{s}} P_t(\vec{s})(s_{k_1} s_{k_2} \ldots s_{k_{2n}}) + D \sum_{x \in \mathrm{Z}} \sum_{\vec{s}} P_t(\vec{\sigma_x})(s_{k_1} s_{k_2} \ldots s_{k_{2n}}) = -4nD\mathrm{E}_t(s_{k_1} s_{k_2} \ldots s_{k_{2n}}).$$

By similar argument, if $x \notin \{k_1, k_2 \ldots, k_{2n}\}$,

$$\sum_{\vec{s}} D\left[\frac{s_x(s_{x-1}+s_{x+1})}{2}\right] P_t(\vec{s})(s_{k_1} s_{k_2} \ldots s_{k_{2n}}) = -\sum_{\vec{s}} D\left[\frac{s_x(s_{x-1}+s_{x+1})}{2}\right] P_t(\vec{\sigma_x})(s_{k_1} s_{k_2} \ldots s_{k_{2n}}).$$

Otherwise, if $x \in \{k_1, k_2, \ldots, k_{2n}\}$,

$$\sum_{\vec{s}} D\left[\frac{s_x(s_{x-1}+s_{x+1})}{2}\right] P_t(\vec{s})(s_{k_1} s_{k_2} \ldots s_{k_{2n}}) = \sum_{\vec{s}} D\left[\frac{s_x(s_{x-1}+s_{x+1})}{2}\right] P_t(\vec{\sigma_x})(s_{k_1} s_{k_2} \ldots s_{k_{2n}})$$

because $s_x^2 = 1$ for any $x \in \mathrm{Z}$ and thus there would be no s_x^2 in the summation.

In summary,

$$\begin{aligned}
& \partial_t \mathrm{E}_t(s_{k_1} s_{k_2} \ldots s_{k_{2n}}) \\
= \; & -4nD\mathrm{E}_t(s_{k_1} s_{k_2} \ldots s_{k_{2n}}) + D \sum_{i=1}^{2n} \sum_{\vec{s}} [s_{k_i}(s_{k_i-1}+s_{k_i+1})] P_t(\vec{s})(s_{k_1} s_{k_2} \ldots s_{k_{2n}}) \\
= \; & -4nD\mathrm{E}_t(s_{k_1} s_{k_2} \ldots s_{k_{2n}}) + D \sum_{i=1}^{2n} \sum_{\vec{s}} P_t(\vec{s}) \left(s_{k_1} \ldots s_{k_{i-1}}(s_{k_i-1}+s_{k_i+1}) s_{k_{i+1}} \ldots s_{k_{2n}}\right) \\
= \; & D \sum_{i=1}^{2n} \sum_{\vec{s}} P_t(\vec{s}) \left(s_{k_1} \ldots s_{k_{i-1}}(s_{k_i-1} - 2s_{k_i} + s_{k_i+1}) s_{k_{i+1}} \ldots s_{k_{2n}}\right) \\
= \; & D \sum_{i=1}^{2n} \mathrm{E}_t \left(s_{k_1} \ldots s_{k_{i-1}}(s_{k_i-1} - 2s_{k_i} + s_{k_i+1}) s_{k_{i+1}} \ldots s_{k_{2n}}\right) \\
= \; & D\Delta \mathrm{E}_t(s_{k_1} s_{k_2} \ldots s_{k_{2n}})
\end{aligned} \tag{3.12}$$

where Δ is the discrete Laplacian defined in (3.6) with respect to all the $2n$ positions k_i and $k_1 < k_2 < \cdots < k_n$.

Since by the property of spins $s_i^2 = 1$, we have the boundary condition

$$\mathrm{E}_t\left(s_{k_1} \dots s_{k_i} s_{k_{i+1}} \dots s_{k_{2n}}\right) = \mathrm{E}_t\left(s_{k_1} \dots s_{k_{i-1}} s_{k_{i+2}} \dots s_{k_{2n}}\right) \text{ if } k_i = k_{i+1}. \tag{3.13}$$

We can then set positions $k_{2i+1} = x_i$ and $k_{2i} = y_i$ for $i = 1, \dots, 2n$. From equations (3.7) and (3.12) we can see that both empty interval probability $P_t\left(\underline{x}, \underline{y}\right)$ and spin correlation $\mathrm{E}_t\left(s_{k_1} s_{k_2} \dots s_{k_{2n}}\right)$ satisfy the heat equation. Furthermore, by comparing equations (3.9), (3.10) and (3.13) we can see that they satisfy the same boundary conditions. Since we have assume both equations satisfy the same initial condition and thus by uniqueness of the heat equation we have proved the two equations are identical. □

3.3 Uniqueness of discrete heat equation in unbounded domain

Lemma 10. *For a function $u : \Omega \times [0, \infty) \to R$, the following*

$$\begin{cases} \Delta u < u_t & \text{in } \Omega \times [0, \infty) \\ u \geq 0 & \text{on } \partial\Omega \times [0, \infty) \cup \Omega \times \{0\} \end{cases}$$

implies

$$u \geq 0 \text{ in } \bar{\Omega} \times [0, \infty)$$

where Δ is the discrete Laplacian operator, u_t is the derivative of u with respect to time and $\Omega \subset Z^n$ is a bounded domain.

Proof Firstly, assume u attains minimum on $\Omega \times [0, T]$ at (x^*, t^*). This can be found since Ω is bounded and u is continuous with respect to time and there is a finite number of spatial grid points in Ω.

Now suppose $u\left(x^*, t^*\right) < 0$, otherwise the proof is done, then there are two cases to consider.

Case one: $t^* \neq T$, then $u_t\left(x^*, t^*\right) = 0$ and $\Delta u\left(x^*, t^*\right) \geq 0$ which contradicts the assumption $\Delta u < u_t$. Therefore $u \geq 0$ in $\bar{\Omega} \times [0, \infty)$.

Case two: $t^* = T$, then $u_t\left(x^*, t^*\right) < 0$ and $\Delta u\left(x^*, t^*\right) \geq 0$ which also contradicts the assumption $\Delta u < u_t$. Therefore $u \geq 0$ in $\bar{\Omega} \times [0, \infty)$. □

Lemma 11. *For a function $u : \Omega \times [0, \infty) \to R$, the following*

$$\begin{cases} \Delta u \le u_t & \text{in } \Omega \times [0,\infty) \\ u \ge 0 & \text{on } \partial\Omega \times [0,\infty) \cup \Omega \times \{0\} \end{cases}$$

implies

$$u \ge 0 \text{ in } \bar{\Omega} \times [0, \infty)$$

where Δ is the discrete Laplacian operator, u_t is the derivative of u with respect to time and $\Omega \subset Z^n$ is a bounded domain.

Proof Define

$$v = u + \epsilon t,$$

where $\epsilon > 0$. Then we have

$$v_t = u_t + \epsilon \ge \Delta u + \epsilon > \Delta u = \Delta v.$$

By Lemma 10, we have

$$v \ge 0 \text{ in } \bar{\Omega} \times [0, \infty).$$

Since ϵ can be arbitrarily small, $u \ge 0$ in $\bar{\Omega} \times [0, \infty)$. $\square$

Lemma 12. *For a bounded function $u : \Omega \times [0, \infty) \to R$, the following*

$$\begin{cases} \Delta u - u_t \le -\delta < 0 & \text{in } \Omega \times [0,\infty) \\ u \ge 0 & \text{on } \partial\Omega \times [0,\infty) \cup \Omega \times \{0\} \end{cases}$$

implies

$$u \ge 0 \text{ in } \bar{\Omega} \times [0, \infty)$$

where Δ is the discrete Laplacian operator, u_t is the derivative of u with respect to time, $\delta > 0$ and $\Omega \subset Z^n$ is an unbounded domain.

Proof Define

$$v = u + \epsilon |x|^2.$$

Then

$$\Delta v - v_t = \Delta u + 2n\epsilon - u_t < 0$$

if we choose ϵ small enough. In particular,

$$\epsilon < \frac{\delta}{2n}.$$

Hence

$$\Delta v < v_t.$$

Since u is bounded, $v \geq 0$ for large enough R such that $|x|^2 \geq R$. We can split domain Ω into two parts: $|x|^2 \geq R$ and $|x|^2 < R$.

For $|x|^2 < R$, we can use Lemma 10 to show that $v \geq 0$.

For $|x|^2 \geq R$, $|x|^2$ dominates and thus $v \geq 0$.

In summary, $v \geq 0$ in $\bar{\Omega} \times [0, \infty)$ and hence $u \geq 0$ in $\bar{\Omega} \times [0, \infty)$ as ϵ is arbitrary. □

Lemma 13. *For a bounded function $u : \Omega \times [0, \infty) \to R$, the following*

$$\begin{cases} \Delta u - u_t \leq 0 & \text{in } \Omega \times [0, \infty) \\ u \geq 0 & \text{on } \partial\Omega \times [0, \infty) \cup \Omega \times \{0\} \end{cases}$$

implies

$$u \geq 0 \text{ in } \bar{\Omega} \times [0, \infty)$$

where Δ is the discrete Laplacian operator, u_t is the derivative of u with respect to time and $\Omega \subset Z^n$ is an unbounded domain.

Proof Define

$$v = u + \delta t.$$

It can be seen that

$$\Delta v - v_t = \Delta u - u_t - \delta < 0.$$

By lemma 12 we know that

$$v \geq 0 \text{ in } \bar{\Omega} \times [0, \infty)$$

and since δ can be arbitrarily small we have

$$u \geq 0 \text{ in } \bar{\Omega} \times [0, \infty).$$

□

Now we can also have a counterpart of lemma 13:

Lemma 14. *For a bounded function $u : \Omega \times [0, \infty) \to R$, the following*

$$\begin{cases} \Delta u - u_t \geq 0 & \text{in } \Omega \times [0, \infty) \\ u \leq 0 & \text{on } \partial\Omega \times [0, \infty) \cup \Omega \times \{0\} \end{cases}$$

implies

$$u \leq 0 \text{ in } \bar{\Omega} \times [0, \infty)$$

where Δ is the discrete Laplacian operator, u_t is the derivative of u with respect to time and $\Omega \subset Z^n$ is an unbounded domain.

Proof Substitute $v = -u$ into lemma 13 then we can immediately get the result.

□

Lemma 13 and lemma 14 are the main tool we will use to prove the main theorem in this session.

With lemma 13 and 14 at our disposal we can now prove the uniqueness of heat equation for a bounded function u in an unbounded domain which is the main theorem of this session.

Theorem 15. *If a bounded function $u : \Omega \times [0, \infty) \to R$ satisfies the heat equation*

$$\Delta u = u_t \text{ in } \Omega \times [0, \infty)$$

and a boundary condition

$$u = f \text{ on } \partial\Omega \times [0, \infty) \cup \Omega \times \{0\}$$

then u is unique.

Proof Suppose there are two bounded functions u and v which satistify heat equation

$$\Delta u = u_t \text{ in } \Omega \times [0, \infty)$$

and

$$\Delta v = v_t \text{ in } \Omega \times [0, \infty)$$

and the same boundary condition

$$u = v = f \text{ on } \partial\Omega \times [0, \infty) \cup \Omega \times \{0\}.$$

Then let $w = u - v$ and we can see

$$\Delta w = w_t \text{ in } \Omega \times [0, \infty)$$

and

$$w = 0 \text{ on } \partial\Omega \times [0, \infty) \cup \Omega \times \{0\}.$$

By lemma 13 and 14 we have

$$w = 0 \text{ in } \bar{\Omega} \times [0, \infty).$$

Therefore we have proved the uniqueness of u. $\square$

3.4 Pfaffian and heat equation

Previously we have obtained the kinetic equation of spin correlation. This suggests another method to investigate the Pfaffian property of spin correlation. Firstly let us investigate the general case of second order linear partial differential equation of Pfaffian.

Lemma 16. *Suppose $Pf(A)$ is a Pfaffian of a $2n \times 2n$ matrix A whose entries are functions of time and positions, i.e.*

$$a_{i,j} = (-1)^{\chi(i<j)} g(x_i, x_j; t)$$

and satisfy the second order linear partial differential equation

$$\begin{aligned}
\partial_t a_{i,j} &= \left[b_i \partial_i^2 + b_j \partial_j^2 + c_i \partial_i + c_j \partial_j + f(x_i) + f(x_j)\right] a_{i,j} \\
&= \left[\sum_{l=1}^{2n} b_l \partial_l^2 + \sum_{m=1}^{2n} c_m \partial_m + f(x_i) + f(x_j)\right] a_{i,j}
\end{aligned}$$

where $\partial_i = \frac{\partial}{\partial_{x_i}}$, b_l and c_m are functions of x_i and $f(x_i)$ is a function of x_i.

Then

$$\partial_t Pf(A) = \left[\sum_{l=1}^{2n} b_l \partial_l^2 + c_l \partial_l + f(x_l)\right] Pf(A).$$

Proof By the definition of Pfaffian it can be expressed as

$$\text{Pf}(A) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \text{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(2i-1),\sigma(2i)}(t)$$

where σ is the symmetric group and $\mathrm{sgn}(\sigma)$ is the signature of σ. Therefore,

$$\begin{aligned}
\partial_t \mathrm{Pf}(A) &= \frac{1}{2^n n!} \sum_{\sigma\in S_{2n}} \mathrm{sgn}(\sigma) \\
&\quad \sum_{j=0}^{n-1} \left[\prod_{i=1}^{j} a_{\sigma(2i-1),\sigma(2i)}(t) \left(\partial_t a_{\sigma(2j-1),\sigma(2j)}(t)\right) \prod_{i=j+2}^{n} a_{\sigma(2i-1),\sigma(2i)}(t) \right] \\
&= \frac{1}{2^n n!} \sum_{\sigma\in S_{2n}} \mathrm{sgn}(\sigma) \\
&\quad \sum_{j=0}^{n-1} \{ \prod_{i=1}^{j} a_{\sigma(2i-1),\sigma(2i)}(t) \\
&\quad [b_{\sigma(2i-1)}\partial^2_{\sigma(2i-1)} + b_{\sigma(2i)}\partial^2_{\sigma(2i)} + \\
&\quad c_{\sigma(2i-1)}\partial_{\sigma(2i-1)} + c_{\sigma(2i)}\partial_{\sigma(2i)} + f\left(x_{\sigma(2j-1)}\right) + f\left(x_{\sigma(2j)}\right)] a_{\sigma(2j-1),\sigma(2j)}(t) \\
&\quad \prod_{i=j+2}^{n} a_{\sigma(2i-1),\sigma(2i)}(t) \} \\
&= \frac{1}{2^n n!} \sum_{\sigma\in S_{2n}} \mathrm{sgn}(\sigma) \left[\sum_{l=1}^{2n} b_l \partial_l^2 + c_l \partial_l + f(x_l) \right] \left(\prod_{i=1}^{n} a_{\sigma(2i-1),\sigma(2i)}(t) \right) \\
&= \left[\sum_{l=1}^{2n} b_l \partial_l^2 + c_l \partial_l + f(x_l) \right] \mathrm{Pf}(A)
\end{aligned}$$

Therefore, the Pfaffian of matrix A whose entries are of this particular form satisfies the above second order linear partial differential equation. $\square$

Corollary 17. *The Pfaffian $Pf(A)$ of a $2n \times 2n$ matrix A satisfies the heat equation*

$$\Delta Pf(A) = \partial_t Pf(A)$$

if its entries are functions of time and positions, i.e.

$$a_{i,j} = (-1)^{\chi(i<j)} g(x_i, x_j; t)$$

and satisfy the heat equation

$$\Delta a_{i,j} = \partial_t a_{i,j}$$

where $\Delta = \sum_{i=1}^{2n} \partial_i^2$.

Proof Let $b_i = 1$, $c_i = 0$ and $f(x_i) = 0$ for all i. $\square$

Lemma 18. *For a Pfaffian of a $2n \times 2n$ anti-symmetric matrix A, denote*

$$Pf(A) = Pf(x_1, \ldots, x_{2n})$$

where $a_{i,j} = (-1)^{\chi(i>j)} f(x_i, x_j)$ for some function f. If $x_i = x_{i+1}$, then

$$Pf(x_1, \ldots, x_{2n}) = a_{i,i+1} Pf(x_1, \ldots, x_{i-1}, x_{i+2}, \ldots, x_{2n}).$$

where $Pf(x_i, \ldots, x_{i-1}, x_{i+2}, \ldots, x_{2n})$ is the Pfaffian of a $(2n-2)\times(2n-2)$ anti-symmetric matrix obtained from A by removing the i-th and $i+1$-th rows and columns.

Proof By the definition of Pfaffian,

$$\mathrm{Pf}(A) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(2i-1),\sigma(2i)}.$$

Consider one of the terms in the summation of the form

$$a_{\sigma(1),\sigma(2)} \cdots a_{i,\sigma(j)} \cdots a_{i+1,\sigma(k)} \cdots a_{\sigma(2n-1),\sigma(2n)}.$$

Suppose there are m a's between $a_{i,\sigma(j)}$ and $a_{i+1,\sigma(k)}$, then it takes $4m+3$ transpositions to obtain

$$a_{\sigma(1),\sigma(2)} \cdots a_{i+1,\sigma(j)} \cdots a_{i,\sigma(k)} \cdots a_{\sigma(2n-1),\sigma(2n)}$$

and thus the sgn function of these two permutations will differ by -1. If $i = i+1$ then these two terms will cancel each other. The same reasoning applies to the terms of the forms

$$a_{\sigma(1),\sigma(2)} \cdots a_{i,\sigma(j)} \cdots a_{\sigma(k),i} \cdots a_{\sigma(2n-1),\sigma(2n)}$$

and

$$a_{\sigma(1),\sigma(2)} \cdots a_{\sigma(j),i} \cdots a_{\sigma(k),i} \cdots a_{\sigma(2n-1),\sigma(2n)}$$

and

$$a_{\sigma(1),\sigma(2)} \cdots a_{\sigma(j),i} \cdots a_{i,\sigma(k)} \cdots a_{\sigma(2n-1),\sigma(2n)}.$$

They are all cancelled by their partners. Only the terms of the form

$$a_{\sigma(1),\sigma(2)} \cdots a_{i,i+1} \cdots a_{\sigma(2n-1),\sigma(2n)}$$

remains and for each permutation σ there are $2n$ of them and therefore

$$\mathrm{Pf}\,(A) = \frac{2n}{2^n n!} \sum_{\sigma \in S_{2n-2}} a_{i,i+1} \mathrm{sgn}\,(\sigma) \prod_{i=1}^{n-1} a_{\sigma(2i-1),\sigma(2i)} = a_{i,i+1} \mathrm{Pf}\,(x_i, \ldots, x_{i-1}, x_{i+2}, \ldots, x_{2n})\,.$$

$\square$

Corollary 19. *If $a_{i,j} = 1$ for $i = j$, then*

$$Pf(x_1, \ldots, x_{2n}) = Pf(x_1, \ldots, x_{i-1}, x_{i+2}, \ldots, x_{2n})\,.$$

The above corollaries can be used as an alternative proof that the spin correlation function is Pfaffian. We just have to show that the boundary conditions and the initial conditions of the correlation functions are the same as that of a particular Pfaffian.

Theorem 20. *The spin correlation $E_t\,(x_1, \ldots, x_{2n})$ is a Pfaffian $Pf(S)$ under the Bernoulli initial condition*

$$|s_k\rangle = \frac{1}{\sqrt{2}}\,(|\downarrow\rangle + |\uparrow\rangle)$$

where S is a skew-symmetric matrix

$$S_{i,j} = (-1)^{\chi(i>j)} E_t\,(s_i s_j)\,.$$

Proof By the resulf in session 3.2 and corollary 17 we know that the spin correlation $\mathrm{E}_t\,(x_1 \cdots x_{2n})$ and the Pfaffian $\mathrm{Pf}\,(S)$ both satisfy the heat equation.

By corollary 19 and the fact that $s_k^2 = 1$ we can see that the spin correlation $\mathrm{E}_t\,(x_1 \cdots x_{2n})$ and the Pfaffian $\mathrm{Pf}\,(S)$ satisfy the same boundary condition.

Since $\mathrm{E}_{t=0}\,(s_{x_1} \cdots s_{x_{2n}}) = \mathrm{E}_{t=0}\left(s_{x_i} s_{x_j}\right) = 0$ for our initial condition, the spin correlation $\mathrm{E}_t\,(x_1 \cdots x_{2n})$ and the Pfaffian $\mathrm{Pf}\,(S)$ satisfy the same initial condition. $\square$

The advantage of this approach is that it is easier to generalise to other initial conditions and to the case of non-zero temperature.

Chapter 4

Pfaffian point process

In this chapter we will show that the n-point correlations $\rho_n(x_1, \ldots, x_n)$ for ARW and CRW have similar format, which can be expressed as a Pfaffian of a $2n \times 2n$ matrix. A random point process possessing this structure is called a Pfaffian point process. The exact definition of Pfaffian and Pfaffian point process were given earlier in section 1.1 and 1.2

We consider CRW/ARW on a one-dimensional discrete lattice Zand are interested in obtaining the exact equation of the correlation function in coalescing/annihilating random walk.

By employing free fermionic operators and empty interval method in the annihilating and coalescing cases respectively, it is found that the correlation functions possess a Pfaffian property.

It is found that the kernel of the Pfaffian is written in terms of the two-point spin correlation function $\mathrm{E}(s_{x_i}s_{x_j})$ in equation (2.28), which only depends on the distance between x_i and x_j. Let us recall the 2-point function has the form:

$$\mathrm{E}_t(s_{x_i}s_{x_j+k}) = 1 - e^{-2t} \oint_{C_\epsilon} \frac{d\lambda}{2\pi i} \frac{\lambda^{-k} - \lambda^k}{1-\lambda} e^{t(\lambda+\lambda^{-1})}$$

In the next two sections we will prove that ARW and CRW are Pfaffian point processes by finding the matrix kernel $K(x, y)$.

4.1 Annihilating case

Theorem 21. *For an initial condition that every site has independent $\frac{1}{2}$ probability being occupied , the correlation function of ARW at zero temperature is Pfaffian:*

$$\rho_n^{ARW}(x_1,\ldots,x_n;t)=\frac{Pf(I-S)}{2^n}=\frac{Pf(K)}{2^n}$$

where I is a $2n\times 2n$ block diagonal matrix with n blocks of $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ on the diagonal and S is a $2n\times 2n$ skew-symmetric matrix defined in Theorem 5. Therefore K has the matrix kernels

$$K(x,y) \;=\; \begin{pmatrix} E_t\left(s_x s_y\right) & E_t\left(s_x s_{y+1}\right)\\ E_t\left(s_{x+1} s_y\right) & E_t\left(s_{x+1} s_{y+1}\right)\end{pmatrix}$$

above the diagonal; and

$$K(x_i,x_j) \;=\; \begin{pmatrix} 0 & 1-E_t\left(s_{x_i} s_{x_j}\right)\\ -1+E_t\left(s_{x_i} s_{x_j}\right) & 0\end{pmatrix}$$

on the diagonal. The blocks below the diagonal can be obtained by the identity $K_{ij}(x,y)=-K_{ji}(y,x)$.

Proof By using lemma 1, we can get

$$\mathrm{Pf}\left(I-S\right)=\sum_J(-1)^{|J|/2}(-1)^{s(J)}\mathrm{Pf}\left(I|_J\right)\mathrm{Pf}\left(S^T|_{J^c}\right).$$

We can observe that $\mathrm{Pf}\left(I|_J\right)=1$ only if $J=\{\ldots,2j-1,2j,\ldots,2k-1,2k,\ldots\},1\leq j,k\leq n$. Otherwise $\mathrm{Pf}\left(I|_J\right)=0$. Denote this type of subset of $\{1,2,\ldots,2n\}$ by $\tilde{J}$. Also we can observe that $(-1)^{|\tilde{J}|/2}(-1)^{s(\tilde{J})}=1$ and $S^T=-S$. Therefore,

$$\begin{aligned}\mathrm{Pf}\left(I-S\right) &= \sum_{\tilde{J}}\mathrm{Pf}\left(S^T|_{\tilde{J}^c}\right)\\ &= 1+\sum_{J_2}\mathrm{Pf}\left(S^T|_{J_2}\right)+\sum_{J_4}\mathrm{Pf}\left(S^T|_{J_4}\right)+\cdots+\mathrm{Pf}\left(S^T\right)\end{aligned}$$

where J_k is obtained by taking k pairs of adjacent columns and rows from the matrix J, i.e. $J_k=\{\ldots,2i_1-1,2i_1,\ldots,2i_2-1,2i_2,\ldots,2i_k-1,2i_k,\ldots\},1\leq i_1\leq\cdots\leq i_k\leq n$

and by equation (1.1),

$$\mathrm{Pf}\,(I-S) = 1 - \sum_{J_2} \mathrm{Pf}\,(S|_{J_2}) + \sum_{J_4} \mathrm{Pf}\,(S|_{J_4}) + \cdots + (-1)^n \mathrm{Pf}\,(S)\,.$$

By Theorem 5,

$$\begin{aligned}
\mathrm{Pf}\,(I-S) &= 1 - \sum_{1\le i\le n} \mathrm{E}_t\,(s_{x_i}s_{x_i+1}) + \sum_{1\le i<j\le n} \mathrm{E}_t\,\left(s_{x_i}s_{x_i+1}s_{x_j}s_{x_j+1}\right) \\
&\quad + \cdots + (-1)^n \mathrm{E}_t\left(\prod_{i=1}^{n} s_{x_i}s_{x_i+1}\right) \\
&= \mathrm{E}_t\left(\prod_{k=1}^{n}\left(1 - s_{x_k}s_{x_k+1}\right)\right)
\end{aligned}$$

Since we can define the domain wall of Glauber model to be our annihilating particles, we have

$$\frac{1 - s_{x_k}s_{x_k+1}}{2} = \delta\,(x_k)\,.$$

Therefore, we have

$$\begin{aligned}
\mathrm{Pf}\,(I-S) &= 2^n \mathrm{E}\left(\prod_{k=1}^{n} \delta\,(x_k)\right) \\
&= 2^n \rho_n^{ARW}\,(x_1,\ldots,x_n;t)\,.
\end{aligned}$$

$\square$

4.1.1 An alternative proof

Another way to prove single time correlation function of annihilating random walk is Pfaffian point process under Bournoulli initial condition, i.e.

Theorem 22.

$$E\left[\prod_{i=1}^{N} n_{z_i}\right] = \left(\frac{-1}{2}\right)^N Pf[K\,(z_i,z_j)]$$

where

$$K\,(z_i,z_j) = \begin{pmatrix} c\,(z_i,z_j) & c\left(z_i,z_j^+\right) - c\,(z_i,z_j) \\ c\left(z_i^+,z_j\right) - c\,(z_i,z_j) & 2c\,(z_i,z_j) - c\left(z_i^+,z_j\right) - c\left(z_i,z_j^+\right) \end{pmatrix}$$

for $z_i < z_j$*, and*

$$K(z_i, z_j) = -\begin{pmatrix} c(z_i, z_j) & c\left(z_i^+, z_j\right) - c(z_i, z_j) \\ c\left(z_i, z_j^+\right) - c(z_i, z_j) & 2c(z_i, z_j) - c\left(z_i^+, z_j\right) - c\left(z_i, z_j^+\right) \end{pmatrix}$$

for $z_i > z_j$*, and*

$$K(z_i, z_j) = \begin{pmatrix} 0 & c\left(z_i, z_i^+\right) - 1 \\ 1 - c\left(z_i, z_i^+\right) & 0 \end{pmatrix}$$

for $z_i = z_j$*, where*

$$c(x, y) = \begin{cases} E(s_x s_y) & \textit{if } x < y \\ -E(s_x s_y) & \textit{if } x > y \\ 1 & \textit{otherwise} \end{cases}.$$

where $x^+ = x + 1$*, under the initial condition that* $P(n_{z_i} = 1) = \frac{1}{2}$ *for all* $i \in Z$*.*

Proof By defining the discrete derivative

$$\partial_{\xi_i} s_{\xi_i} = s_{\xi_i^+} - s_{\xi_i}$$

we have

$$n_{z_i} = \frac{1 - s_{z_i^+} s_{z_i}}{2} = \frac{-1}{2} s_{z_i}\left(s_{z_i^+} - s_{z_i}\right) = \left(\frac{-1}{2}\right) s_{z_i}\left(\partial_{\xi_i} s_{\xi_i}|_{\xi_i = z_i}\right)$$

where $\xi_i \geq z_i$.

Therefore,

$$\begin{aligned} \mathrm{E}\left[\prod_{i=1}^{N} n_{z_i}\right] &= \left(\frac{-1}{2}\right)^N \mathrm{E}\left[\prod_{i=1}^{N} s_{z_i} \partial_{\xi_i} s_{\xi_i}\right] |_{\xi_i = z_i} \\ &= \left(\frac{-1}{2}\right)^N \prod_{i=1}^{N} \partial_{\xi_i} \mathrm{E}\left[\prod_{i=1}^{N} s_{z_i} s_{\xi_i}\right] |_{\xi_i = z_i} \end{aligned}$$

where $z_1 < \xi_1 \leq \cdots \leq z_i < \xi_i \leq z_{i+1} < \cdots \leq z_n < \xi_n$.

And we know that $\mathrm{E}\left[\prod_{i=1}^{N} s_{z_i} s_{\xi_i}\right]$ is a $2n \times 2n$ anti-symmetric matrix which has n^2 2×2 blocks $K_{i,j}$:

$$K_{i,j} = \begin{pmatrix} c(z_i, z_j) & c(z_i, \xi_j) \\ c(\xi_i, z_j) & c(\xi_i, \xi_j) \end{pmatrix}$$

for $i < j$. By using $K_{j,i} = -\left(K_{i,j}\right)^T$ we get $K_{i,j}$for $i > j$ and

$$K_{i,i} = \begin{pmatrix} 0 & c\left(z_i, \xi_i\right) \\ -c\left(z_i, \xi_i\right) & 0 \end{pmatrix}$$

for the ordering $z_1 < \xi_1 \leq \cdots \leq z_i < \xi_i \leq z_{i+1} < \cdots \leq z_n < \xi_n$.

For other orderings of z_i, for example $z_1 < z_2 < \cdots < z_n < z_{n-1}$, we know that it involves even number of transpositions to go from $z_1 < \xi_1 \leq \cdots \leq z_i < \xi_i \leq z_{i+1} < \cdots \leq z_n < \xi_n$ to $z_1 < \xi_1 \leq \cdots \leq z_i < \xi_i \leq z_{i+1} < \cdots \leq z_n < \xi_n \leq z_{n-1} < \xi_{n-1}$ and therefore the same anti-symmetric matrix can represent the spin correlation $\mathrm{E}\left[\prod_{i=1}^N s_{z_i} s_{\xi_i}\right]$.

As we can see the derivatives only apply to the second column and second row of the 2×2 block, we have

$$\partial_{\xi_i} \partial_{\xi_j} K_{i,j} = \begin{pmatrix} c\left(z_i, z_j\right) & c\left(z_i, \xi_j^+\right) - c\left(z_i, \xi_j\right) \\ c\left(\xi_i^+, z_j\right) - c\left(\xi_i, z_j\right) & 2c\left(\xi_i, \xi_j\right) - c\left(\xi_i^+, \xi_j\right) - c\left(\xi_i, \xi_j^+\right) \end{pmatrix}$$

where we set $c\left(\xi_i + 1, \xi_j + 1\right) = c\left(\xi_i, \xi_j\right)$ since the initial condition has translational symmetry.

By setting $\xi_i = z_i$ for all i we have thus proved the theorem. $\square$

Remark The kernel we got by using decoposition of Pfaffian matrices is

$$K\left(z_i, z_j\right) = -\begin{pmatrix} c\left(z_i, z_j\right) & c\left(z_i, z_j^+\right) \\ c\left(z_i^+, z_j\right) & c\left(z_i, z_j\right) \end{pmatrix}$$

for $z_i < z_j$, and

$$K\left(z_i, z_j\right) = -\begin{pmatrix} 0 & c\left(z_i, z_i^+\right) - 1 \\ 1 - c\left(z_i, z_i^+\right) & 0 \end{pmatrix}$$

for $z_i = z_j$. It can be shown that

$$\begin{aligned} &\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c\left(z_i, z_j\right) & c\left(z_i, z_j^+\right) - c\left(z_i, z_j\right) \\ c\left(z_i^+, z_j\right) - c\left(z_i, z_j\right) & 2c\left(z_i, z_j\right) - c\left(z_i^+, z_j\right) - c\left(z_i, z_j^+\right) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \\ = &\begin{pmatrix} 0 & c\left(z_i, z_i^+\right) - 1 \\ 1 - c\left(z_i, z_i^+\right) & 0 \end{pmatrix} \end{aligned}$$

and

$$\begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}\begin{pmatrix}0 & c\left(z_i, z_i^+\right)-1\\1-c\left(z_i, z_i^+\right) & 0\end{pmatrix}\begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}$$
$$= \begin{pmatrix}0 & c\left(z_i, z_i^+\right)-1\\1-c\left(z_i, z_i^+\right) & 0\end{pmatrix}$$

and also

$$det\begin{pmatrix}1 & 0\\1 & 1\end{pmatrix} = det\begin{pmatrix}1 & 0\\1 & 1\end{pmatrix} = 1$$

.

So the two representations of the Pfaffian kernels are equivalent to each other.

4.2 Coalescing case

To derive the n-point correlation function from the empty interval probability $P_t\left(\underline{x}, \underline{y}\right)$ we have to use the technique of dual lattices again but here our definition is a bit different from the one in the previous chapter. We define the particles on the lattice $Z+0.5$ so that the resulting equation will have a format similar to that of ARW.

Definition 5. *The empty interval probability, denote by* $P_t\left[\Omega_{x_1^-,y_1^+}\cap\cdots\cap\Omega_{x_n^-,y_n^+}\right]$*, is the probability that the positions* $\{x^*\in Z : x_i^- < x^* < y_i^+\}, i=1,\ldots,n$ *are empty at time* t*, where* $\Omega_{x_i^-,y_i^+}$ *denotes the event that the positions* $\{x^*\in Z : x_i^- < x^* < y_i^+\}$ *are empty and* x_k^- *and* y_k^+ *denote* $x_k - 0.5$ *and* $y_k + 0.5$ *respectively.*

For example, $P_t\left[\Omega_{1^-,3^+}\right]$ is the probability that the positions $x=\{1,2,3\}$ are not occupied by particles at time t. And $P_t\left(\Omega_{x^-,x^+}\right)$ stands for the probability that position x is occupied, or in other words, the particle density of x, at time t. Also it is trivially true that $P_t\left[\Omega_{x,x}\right]=1$.

We will also need the following theorem, which is a special case of Theorem 9.

Theorem 23. *The probability* $P_t\left[\Omega_{x_1,x_1+1}\cap\cdots\cap\Omega_{x_n,x_n+1}\right]$ *and the spin correlation function* $E\left(s_1, s_1+1,\ldots,s_n,s_n+1\right)$ *both satisfy the heat equation and the same set of boundary conditions and hence are identical equations by the uniqueness theorem of the discrete heat equation, i.e.*

$$P_t\left(\Omega_{x_1,x_1+1}\cap\cdots\cap\Omega_{x_n,x_n+1}\right) = E_t\left(s_{x_1}s_{x_1+1}\ldots s_{x_n}s_{x_n+1}\right).$$

We are now ready to calculate the correlation function $\rho_n^{CRW}\left(x_1,\ldots,x_n;t\right)$.

Theorem 24. *For an initial condition such that every site is occupied, the correlation function of CRW is Pfaffian:*

$$\rho_n^{CRW}(x_1, \ldots, x_n; t) = 2^n Pf(I - S) = 2^n Pf(K)$$

where the definition of the matrices I, S and K are the same as before.

Proof The correlation function can be expressed by the probability

$$\rho_n^{CRW}(x_1, \ldots, x_n; t) = P_t\left[\left(\Omega \setminus \Omega_{x_1^-, x_1^+}\right) \cap \cdots \cap \left(\Omega \setminus \Omega_{x_n^-, x_n^+}\right)\right],$$

where Ω is the whole probability space.

Since the probability of an event can be expressed by the expectation of an indicator function of the event, we have

$$\rho_n^{CRW}(x_1, \ldots, x_n; t) = \mathrm{E}_t\left(I\left[\left(\Omega \setminus \Omega_{x_1^-, x_1^+}\right) \cap \cdots \cap \left(\Omega \setminus \Omega_{x_n^-, x_n^+}\right)\right]\right).$$

We can decompose the events in the indicator function by the following identities:

$$I(\Omega \setminus A) = 1 - I(A) \tag{4.1}$$
$$I(A \cap B) = I(A) I(B). \tag{4.2}$$

Therefore,

$$\begin{aligned} I\left[\left(\Omega \setminus \Omega_{x_1^-, x_1^+}\right) \cap \cdots \cap \left(\Omega \setminus \Omega_{x_n^-, x_n^+}\right)\right] &= \prod_{i=1}^{n} I[\Omega \setminus \Omega_{x_i^-, x_i^+}] \\ &= \prod_{i=1}^{n} 1 - I[\Omega_{x_i^-, x_i^+}]. \end{aligned}$$

Therefore we have

$$\begin{aligned} \rho_n^{CRW}(x_1, \ldots, x_n; t) &= \mathrm{E}_t\left(I\left[\left(\Omega \setminus \Omega_{x_1^-, x_1^+}\right) \cap \cdots \cap \left(\Omega \setminus \Omega_{x_n^-, x_n^+}\right)\right]\right) \\ &= \mathrm{E}_t\left(\prod_{i=1}^{n} 1 - I[\Omega_{x_i^-, x_i^+}]\right). \end{aligned}$$

By (4.2),

$$\begin{aligned} & \mathrm{E}_t\left(I\left[\Omega_{x_i^-,x_i^+}\right]\dots I\left[\Omega_{x_j^-,x_j^+}\right]\right) \\ = \; & \mathrm{E}_t\left(I\left[\Omega_{x_i^-,x_i^+}\cap\dots\cap\Omega_{x_j^-,x_j^+}\right]\right) \\ = \; & P_t\left(\Omega_{x_i^-,x_i^+}\cap\dots\cap\Omega_{x_j^-,x_j^+}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} & \rho_n^{CRW}(x_1,\dots,x_n;t) \\ = \; & \mathrm{E}_t\left(\prod_{i=1}^n 1 - I\left[\Omega_{x_i^-,x_i^+}\right]\right) \\ = \; & 1-\sum_{i=1}^n P_t\left[\Omega_{x_i^-,x_i^+}\right]+\sum_{1\le i<j\le n} P_t\left[\Omega_{x_i^-,x_i^+,x_j^-,x_j^+}\right]+\dots+(-1)^n P_t\left[\Omega_{x_1^-,x_1^+,\dots,x_n^-,x_n^+}\right]. \end{aligned}$$

By the Theorem 23 we have

$$P_t\left(\Omega_{x_i^-,x_i^+}\cap\dots\cap\Omega_{x_j^-,x_j^+}\right)=\mathrm{E}_t\left(s_{x_i^-}s_{x_i^+}\dots s_{x_j^-}s_{x_j^+}\right).$$

Therefore,

$$\begin{aligned} & \rho_n^{CRW}(x_1,\dots,x_n;t) \\ = \; & 1-\sum_{i=1}^n \mathrm{E}_t\left(s_{x_i^-}s_{x_i^+}\right)+\sum_{1\le i<j\le n}\mathrm{E}_t\left(s_{x_i^-}s_{x_i^+}s_{x_j^-}s_{x_j^+}\right)+\dots+(-1)^n\mathrm{E}_t\left(s_{x_1^-}s_{x_1^+}\dots s_{x_n^-}s_{x_n^+}\right). \end{aligned}$$

Since $\mathrm{E}_t\left(s_{x_1^-}s_{x_1^+}\dots s_{x_n^-}s_{x_n^+}\right)$ is a Pfaffian of two-point functions and the two-point functions depends on the absolute value of the distances, we can shift all the coordintates by 0.5, i.e.

$$\mathrm{E}_t\left(s_{x_1^-}s_{x_1^+}\dots s_{x_n^-}s_{x_n^+}\right)=\mathrm{E}_t\left(s_{x_1}s_{x_1+1}\dots s_{x_n}s_{x_n+1}\right).$$

Hence,

$$\begin{aligned}
& \rho_n^{CRW}\left(x_1,\ldots,x_n;t\right) \\
= \; & 1-\sum_{i=1}^{n}\mathrm{E}_t\left(s_{x_i}s_{x_i+1}\right)+\sum_{1\leq i<j\leq n}\mathrm{E}_t\left(s_{x_i}s_{x_i+1}s_{x_j}s_{x_j+1}\right)+\cdots+(-1)^n\mathrm{E}_t\left(s_{x_1}s_{x_1+1}\ldots s_{x_n}s_{x_n+1}\right) \\
= \; & \mathrm{E}_t\left(\prod_{n=1}^{n}1-s_{x_i}s_{x_i+1}\right) \\
= \; & 2^n\mathrm{E}_t\left(\prod_{k=1}^{n}\delta\left(x_k\right)\right) \\
= \; & \mathrm{Pf}\left(I-S\right).
\end{aligned}$$

□

Remark From Theorem 24 and Theorem 21 we can see that for our special initial conditions the particle correlations of the two systems are related by

$$\rho_n^{CRW}\left(x_1,\ldots,x_n;t\right)=2^n\rho_n^{ARW}\left(x_1,\ldots,x_n;t\right).$$

From now on we may refer to the initial conditions of ARW and CRW in this chapter as maximal entrance law.

Chapter 5

Generalisation of models

5.1 Spontaneous creation of pairs of particles in ARW

In the previous chapters we discussed the case in which the "temperature" of the Glauber model is zero, which corresponds to the case in which the spins tend to align with their neighbours. If we consider the case of non-zero finite temperature, then there will be spontaneous disalignment of spins and this corresponds to spontaneous creation of pair of particles in ARW. Surprisingly, with this spontaneous creation of particles in the system, the Pfaffian property is still preserved.

Firstly we will prove by fermionic representation.

5.1.1 Proof by fermionic representation

Theorem 25. *For an initial condition that every site has independent $\frac{1}{2}$ probability being occupied , the correlation function of ARW at non-zero temperature is Pfaffian:*

$$\rho_n^{ARW}(x_1,\ldots,x_n;t) = \frac{Pf(I-S)}{2^n} = \frac{Pf(K)}{2^n}$$

where I is a $2n \times 2n$ block diagonal matrix with n blocks of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on the diagonal and S is a $2n \times 2n$ skew-symmetric matrix with the matrix kernel

$$\begin{aligned} K(x_i,x_j) &= (-1)^{\chi(i>j)} \begin{pmatrix} E_t\left(s_{x_i}s_{x_j}\right) & E_t\left(s_{x_i}s_{x_j+1}\right) \\ E_t\left(s_{x_i+1}s_{x_j}\right) & E_t\left(s_{x_i+1}s_{x_j+1}\right) \end{pmatrix} \\ &= (-1)^{\chi(i>j)} \begin{pmatrix} r_{|x_i-x_j|}(t) & r_{|x_i-x_j|+1}(t) \\ r_{|x_i-x_j|+1}(t) & r_{|x_i-x_j|}(t) \end{pmatrix} \end{aligned}$$

for $i \neq j$*;*

$$
\begin{aligned}
K(x_i, x_j) &= \begin{pmatrix} 0 & 1 - E_t\left(s_{x_i} s_{x_j}\right) \\ -1 + E_t\left(s_{x_i} s_{x_j}\right) & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 - r_{|x_i - x_j|}\left(t\right) \\ -1 + r_{|x_i - x_j|}\left(t\right) & 0 \end{pmatrix}
\end{aligned}
$$

for $i = j$*.*

Proof To investigate this case, we need to change the transition rate ω of the spin s_k at position k. In particular, γ is no longer assumed to be $\frac{-1}{2}$. In fact, $\frac{-1}{2} \leq \gamma \leq 0$. But as we can see in equation (2.17) and in the derivation of (2.23), the assumption that $\gamma = \frac{-1}{2}$ is not used and therefore the spin correlation (2.23) would remain the same form and the only difference is that $\mathfrak{L}$ and hence $\psi\left(t\right)$ is different.

Therefore, lemma 3 still holds as the new $\mathfrak{L}$ is still quadratic in terms of ψ and $\psi^{\dagger}$ and hence lemma 4 still holds, which means the commutation structure of the ψ operators is still preserved. The only difference is that the two point function in this case is a new one at non-zero temperature. Thus the argument of Theorem 5 goes through exactly as before, i.e. we still have

$$
\mathrm{E}_t\left(s_{x_1} \cdots s_{x_n}\right) = \mathrm{Pf}\left(S\right)
$$

where S is a skew-symmetric matrix

$$
S_{i,j} = (-1)^{\chi(i<j)} \mathrm{E}_t\left(s_i s_j\right),
$$

where χ is an indicator function.

With this at our disposal we can replicate the proof of Theorem 21 exactly as before and thus prove that at non-zero temperature ARW is also a Pfaffian point process.

$\square$

5.1.2 Alternative approach

Kinetic equation of spin correlation at non-zero temperature

An alternative way to obtain the same result is to observe that the kinetic equation of spin correlation at non-zero temperature will preserve Pfaffian. Thus the spin correlation is still a Pfaffian at non-zero temperature and thus ARW with immigration is also a Pfaffian.

Lemma 26. *For non-zero temperature the spin correlation $E_t(s_{k_1}\cdots s_{k_{2n}})$ satisfies the following kinetic equation:*

$$\partial_t E_t = \left[-2\gamma D\Delta - 2D(2n)(1+2\gamma)\right] E_t$$

where $\Delta = \sum_{i=1}^{2n} \partial_{k_i}^{-}\partial_{k_i}^{+}$, as defined before in (3.6).

Proof Following the proof of Theorem 9 in section 3.2 we can see

$$\partial_t P_t(\vec{s}) = -D\sum_{x\in\mathrm{Z}}\left[1+\gamma s_x(s_{x-1}+s_{x+1})\right]P_t(\vec{s}) + D\sum_{x\in\mathrm{Z}}\left[1-\gamma s_x(s_{x-1}+s_{x+1})\right]P_t(\vec{\sigma_x}).$$

Therefore,

$$\begin{aligned}
&\partial_t \mathrm{E}_t(s_{k_1}s_{k_2}\ldots s_{k_{2n}}) \\
= \;& -\sum_{\vec{s}}\sum_{x\in\mathrm{Z}} D\left[1+\gamma s_x(s_{x-1}+s_{x+1})\right]P_t(\vec{s})(s_{k_1}s_{k_2}\ldots s_{k_{2n}}) \\
&+\sum_{\vec{s}}\sum_{x\in\mathrm{Z}} D\left[1-\gamma s_x(s_{x-1}+s_{x+1})\right]P_t(\vec{\sigma_x})(s_{k_1}s_{k_2}\ldots s_{k_{2n}}).
\end{aligned}$$

If $x \notin \{k_1, k_2, \ldots, k_{2n}\}$,

$$D\sum_{\vec{s}} P_t(\vec{s})(s_{k_1}s_{k_2}\ldots s_{k_{2n}}) = D\sum_{\vec{s}} P_t(\vec{\sigma_x})(s_{k_1}s_{k_2}\ldots s_{k_{2n}}).$$

Otherwise, if $x \in \{k_1, k_2, \ldots, k_{2n}\}$,

$$-D\sum_{\vec{s}} P_t(\vec{s})(s_{k_1}s_{k_2}\ldots s_{k_{2n}}) = D\sum_{\vec{s}} P_t(\vec{\sigma_x})(s_{k_1}s_{k_2}\ldots s_{k_{2n}}).$$

Therefore,

$$-D\sum_{x\in\mathrm{Z}}\sum_{\vec{s}} P_t(\vec{s})(s_{k_1}s_{k_2}\ldots s_{k_{2n}}) + D\sum_{x\in\mathrm{Z}}\sum_{\vec{s}} P_t(\vec{\sigma_x})(s_{k_1}s_{k_2}\ldots s_{k_{2n}}) = -4nD\mathrm{E}_t(s_{k_1}s_{k_2}\ldots s_{k_{2n}}).$$

By similar argument, if $x \notin \{k_1, k_2 \ldots, k_{2n}\}$,

$$\sum_{\vec{s}} D\gamma s_x(s_{x-1}+s_{x+1})P_t(\vec{s})(s_{k_1}s_{k_2}\ldots s_{k_{2n}}) = -\sum_{\vec{s}} D\gamma s_x(s_{x-1}+s_{x+1})P_t(\vec{\sigma_x})(s_{k_1}s_{k_2}\ldots s_{k_{2n}}).$$

Otherwise, if $x \in \{k_1, k_2, \ldots, k_{2n}\}$,

$$\sum_{\vec{s}} D\gamma s_x \left(s_{x-1} + s_{x+1}\right) P_t\left(\vec{s}\right)\left(s_{k_1} s_{k_2} \ldots s_{k_{2n}}\right) = \sum_{\vec{s}} D\gamma s_x \left(s_{x-1} + s_{x+1}\right) P_t\left(\vec{\sigma_x}\right)\left(s_{k_1} s_{k_2} \ldots s_{k_{2n}}\right).$$

Therefore,

$$\begin{aligned}
& \partial_t \mathrm{E}_t\left(s_{k_1} s_{k_2} \ldots s_{k_{2n}}\right) \\
= \; & -4nD\mathrm{E}_t\left(s_{k_1} s_{k_2} \ldots s_{k_{2n}}\right) - D\sum_{i=1}^{2n}\sum_{\vec{s}} 2\gamma P_t\left(\vec{s}\right)\left(s_{k_1} \ldots s_{k_{i-1}}\left(s_{k_i-1} + s_{k_i+1}\right) s_{k_{i+1}} \ldots s_{k_{2n}}\right) \\
= \; & D\sum_{i=1}^{2n}\sum_{\vec{s}} P_t\left(\vec{s}\right)\left(-2\gamma\right)\left(s_{k_1} \ldots s_{k_{i-1}}\left(s_{k_i-1} + s_{k_i+1}\right) s_{k_{i+1}} \ldots s_{k_{2n}}\right) - 2\left(s_{k_1} \ldots s_{k_{i-1}} s_{k_i} s_{k_{i+1}} \ldots s_{k_{2n}}\right) \\
= \; & D\sum_{i=1}^{2n}\sum_{\vec{s}} P_t\left(\vec{s}\right)\left(-2\right)\left\{s_{k_1} \ldots s_{k_{i-1}}\left[\gamma\left(s_{k_i-1} + s_{k_i+1}\right) + s_{k_i}\right] s_{k_{i+1}} \ldots s_{k_{2n}}\right\} \\
= \; & D\sum_{i=1}^{2n}\sum_{\vec{s}} P_t\left(\vec{s}\right)\left(-2\right)\left\{s_{k_1} \ldots s_{k_{i-1}}\left[\gamma\left(s_{k_i-1} + s_{k_i+1} - 2s_{k_i}\right) + \left(1-2\gamma\right) s_{k_i}\right] s_{k_{i+1}} \ldots s_{k_{2n}}\right\} \\
= \; & \left[-2\gamma D\Delta - 2D\left(2n\right)\left(1+2\gamma\right)\right] E_t
\end{aligned} \tag{5.1}$$

□

By Lemma 16 the partial differential equation satisfied by the spin correlation at non-zero temperature will also be satisfied by a Pfaffian. The boundary condition and initial condition will remain the same as they are independent of the dynamics of the system. For the spin system they depend only on the property that $s^2 = 1$ and for the Pfaffian they arise from the structure of the anti-symmetric matrix.

So if we can prove that the uniquness theorem of the new discrete partial differential equation then we can prove the new spin correlation is a Pfaffian and hence ARW with spontaneous creation of pairs of particles is a Pfaffian point process.

This will be shown below.

Uniqueness of the kinetic equation $\partial_t u = \left(A\Delta - B\right) u$

This section imitates the proofs in section 3.3 to prove that the uniqueness of the bounded equation u in an unbounded domain which satisfies the kinetic equation

$$\partial_t u = \left(A\Delta - B\right) u.$$

Lemma 27. *For a function $u : \Omega \times [0, \infty) \to R$, the following*

$$\begin{cases} A\Delta u - Bu \leq u_t & \textit{in } \Omega \times [0, \infty) \\ u \geq 0 & \textit{on } \partial\Omega \times [0, \infty) \cup \Omega \times \{0\} \end{cases}$$

implies

$$u \geq 0 \textit{ in } \bar{\Omega} \times [0, \infty)$$

where A and B are positive real numbers, Δ is the discrete Laplacian operator, u_t is the derivative of u with respect to time and $\Omega \subset Z^n$ is a bounded domain.

Proof Firstly, assume that u attains minimum on $\Omega \times [0, T]$ at (x^*, t^*). This can be found since Ω is bounded and u is continuous with respect to time and there is a finite number of spatial grid points in Ω.

Now suppose $u(x^*, t^*) < 0$, otherwise the proof is done

Since $u_t(x^*, t^*) \leq 0$, but $\Delta u(x^*, t^*) \geq 0$ and hence $A\Delta u - Bu > 0$, this leads to the contradiction to the assumption that $A\Delta u - Bu \leq u_t$. Therefore $u \geq 0$ in $\bar{\Omega} \times [0, \infty)$.

□

Lemma 28. *For a bounded function $u : \Omega \times [0, \infty) \to R$, the following*

$$\begin{cases} A\Delta u - Bu - u_t \leq -\delta < 0 & \textit{in } \Omega \times [0, \infty) \\ u \geq 0 & \textit{on } \partial\Omega \times [0, \infty) \cup \Omega \times \{0\} \end{cases}$$

implies

$$u \geq 0 \textit{ in } \bar{\Omega} \times [0, \infty)$$

where A and B are positive real numbers, Δ is the discrete Laplacian operator, u_t is the derivative of u with respect to time, $\delta > 0$ and $\Omega \subset Z^n$ is an unbounded domain.

Proof Define

$$v = u + \epsilon |x|^2.$$

Then

$$A\Delta v - Bv - v_t = A\Delta u - Bu - u_t + \epsilon \left(2nA - |x|^2\right) < 0$$

if we choose ϵ small enough. In particular,

$$\epsilon < \frac{\delta}{2nA}.$$

Hence

$$A\Delta v - Bv < v_t.$$

Since u is bounded, $v \geq 0$ for large enough R such that $|x|^2 \geq R$. We can split domain Ω into two parts: $|x|^2 \geq R$ and $|x|^2 < R$.

For $|x|^2 < R$, we can use Lemma 27 to show that $v \geq 0$.

For $|x|^2 \geq R$, $|x|^2$ dominates and thus $v \geq 0$.

In summary, $v \geq 0$ in $\bar{\Omega} \times [0, \infty)$ and hence $u \geq 0$ in $\bar{\Omega} \times [0, \infty)$ as ϵ is arbitrary. $\square$

Lemma 29. *For a bounded function $u : \Omega \times [0, \infty) \to R$, the following*

$$\begin{cases} A\Delta u - Bu - u_t \leq 0 & \text{in } \Omega \times [0, \infty) \\ u \geq 0 & \text{on } \partial\Omega \times [0, \infty) \cup \Omega \times \{0\} \end{cases}$$

implies

$$u \geq 0 \text{ in } \bar{\Omega} \times [0, \infty)$$

where A and B are positive real numbers, Δ is the discrete Laplacian operator, u_t is the derivative of u with respect to time, $\delta > 0$ and $\Omega \subset Z^n$ is an unbounded domain.

Proof Define

$$v = u + \delta t.$$

It can be seen that

$$A\Delta v - Bv - v_t = A\Delta u - Bu - u_t - \delta\,(1 + Bt) < 0.$$

By lemma 28 we know that

$$v \geq 0 \text{ in } \bar{\Omega} \times [0, \infty)$$

and since δ can be arbitrarily small we have

$$u \geq 0 \text{ in } \bar{\Omega} \times [0, \infty).$$

$\square$

Lemma 30. *For a bounded function $u : \Omega \times [0, \infty) \to R$, the following*

$$\begin{cases} A\Delta u - Bu - u_t \geq 0 & \text{in } \Omega \times [0, \infty) \\ u \leq 0 & \text{on } \partial\Omega \times [0, \infty) \cup \Omega \times \{0\} \end{cases}$$

implies

$$u \leq 0 \text{ in } \bar{\Omega} \times [0, \infty)$$

where A and B are positive real numbers, Δ is the discrete Laplacian operator, u_t is the derivative of u with respect to time, $\delta > 0$ and $\Omega \subset Z^n$ is an unbounded domain.

Proof Substitute $v = -u$ into lemma 29 then we can immediately get the result. $\square$

With lemma 29 and 30 at our disposal we can now prove the uniqueness of the kinetic equation $\partial_t u = (A\Delta - B)\, u$ for a bounded function u in an unbounded domain which is the main theorem of this session.

Theorem 31. *If a bounded function $u : \Omega \times [0, \infty) \to R$ satisfies the kinetic equation*

$$\partial_t u = (A\Delta - B)\, u \text{ in } \Omega \times [0, \infty)$$

and a boundary condition

$$u = f \text{ on } \partial\Omega \times [0, \infty) \cup \Omega \times \{0\}$$

then u is unique.

Proof Suppose there are two bounded functions u and v which satistify heat equation

$$\Delta u = u_t \text{ in } \Omega \times [0, \infty)$$

and

$$\Delta v = v_t \text{ in } \Omega \times [0, \infty)$$

and the same boundary condition

$$u = v = f \text{ on } \partial\Omega \times [0, \infty) \cup \Omega \times \{0\}.$$

Then let $w = u - v$ and we can see

$$\Delta w = w_t \text{ in } \Omega \times [0, \infty)$$

and

$$w = 0 \text{ on } \partial\Omega \times [0, \infty) \cup \Omega \times \{0\}.$$

By lemma 29 and 30 we have

$$w = 0 \text{ in } \bar{\Omega} \times [0, \infty).$$

Therefore we have proved the uniqueness of u. □

5.1.3 The steady state of correlation function of ARW

As time goes by the number of particles in ARW decreses due to annihilation. But now as we have introduced the creation of particles, there might be a balance between annihilation and creation and hence a steady state might be obtained. In this section we would like to investigate the form of the steady state.

Lemma 32. *For a $2n \times 2n$ skew-symmetric matrix A of the form*

$$A_{i,j} = (-1)^{\chi(i>j)}\, \eta^{|x_j - x_i|}$$

where $x_j > x_i$ for $j > i$, denote

$$A = A_{x_1,\ldots,x_{2n}}.$$

Then

$$Pf\left(A_{x_1,\ldots,x_k,\ldots,x_{2n}}\right) = \eta^{\sum_{i=1}^{n}(x_{2i} - x_{2i-1})} = \eta^{\sum_{i=1}^{2n}(-1)^i x_i}.$$

Proof Before we proceed it is useful to know the following lemma:

Lemma 33.

$$Pf(A) = \sum_{i=2}^{2n} (-1)^i\, a_{1i} Pf\left(A_{\hat{1}\hat{i}}\right)$$

where $A_{\hat{1}\hat{i}}$ is the matrix obtained from A with both first and i-th column and row removed.

We prove by induction. For $n = 1$ it is obvious since

$$\mathrm{Pf}\,(x_1, x_2) = \eta^{x_2 - x_1}.$$

Assume the statement is true for $2n-2$, then by lemma 33

$$\begin{aligned}
\mathrm{Pf}\left(A_{x_1,\ldots,x_{2n}}\right) &= \sum_{i=2}^{2n}(-1)^i a_{1i}\mathrm{Pf}\left(A_{\hat{1}\hat{i}}\right)\\
&= \sum_{i=2}^{2n}(-1)^i \eta^{x_i-x_1}\mathrm{Pf}\left(A_{x_2,\ldots,x_{i-1},x_{i+1},\ldots,x_{2n}}\right)\\
&= \eta^{\sum_{i=1}^{n}(x_{2i}-x_{2i-1})}+\sum_{i=3}^{2n}(-1)^i \eta^{x_i-x_1}\eta^{\sum_{j=2}^{i-1}(-1)^{j-1}x_j}\eta^{\sum_{j=i+1}^{2n}(-1)^j x_j}.
\end{aligned}$$

Observe that the i-th term in the summation is

$$(-1)^i \eta^{x_i-x_1}\eta^{\sum_{j=2}^{i-1}(-1)^{j-1}x_j}\eta^{\sum_{j=i+1}^{2n}(-1)^j x_j}=(-1)^i \eta^{-x_1}\eta^{x_i}\eta^{(-1)^{i+1}x_{i+1}}\eta^{\sum_{j=2}^{i-1}(-1)^{j-1}x_j}\eta^{\sum_{j=i+2}^{2n}(-1)^j x_j}$$

while the $i+1$-th term is

$$(-1)^{i+1} \eta^{x_{i+1}-x_1}\eta^{\sum_{j=2}^{i}(-1)^{j-1}x_j}\eta^{\sum_{j=i+2}^{2n}(-1)^j x_j}=(-1)^{i+1} \eta^{-x_1}\eta^{x_i}\eta^{(-1)^{i-1}x_{i+1}}\eta^{\sum_{j=2}^{i-1}(-1)^{j-1}x_j}\eta^{\sum_{j=i+2}^{2n}(-1)^j x_j}.$$

So they cancel each other and therefore the summation is zero and hence the lemma is proved by the principle of mathematical induction. □

Theorem 34. *The steady state of annihilating random walk is Bernoulli.*

Proof From Glauber's paper, the steady state of two point function of spins is

$$\mathrm{E}_{t\to\infty}\left(s_k s_{k+n}\right)=\eta^n=\left[\tanh\left(\frac{J}{kT}\right)\right]^n.$$

Now consider the large-time correlation function for ARW

$$\begin{aligned}
\mathrm{E}_{t\to\infty}\left(\prod_{i=1}^{n}n_i\right) &= \frac{1}{2^n}\mathrm{E}_{t\to\infty}\left(\prod_{i=1}^{n}\left(1-s_i s_{i+1}\right)\right)\\
&= \frac{1}{2^n}\{1-\sum_{i=1}^{n}\mathrm{E}_{t\to\infty}\left(s_i s_{i+1}\right)+\sum_{i<j}^{n}\mathrm{E}_{t\to\infty}\left(s_i s_{i+1}s_j s_{j+1}\right)\\
&\quad +\cdots+(-1)^n\mathrm{E}_{t\to\infty}\left(s_1 s_{1+1}\cdots s_n s_{n+1}\right)\}.
\end{aligned}$$

Here we denote s_{i+1} as the spin at x_i+1 for brevity of notation.

As we have shown in Theorem 5 that the spin correlation $\mathrm{E}_t\left(s_i s_{i+1}\cdots s_j s_{j+1}\right)$ is a Pfaffian of two-point functions at any time t given the initial condition, we have

$$\begin{aligned}
\mathrm{E}_{t\to\infty}\left(\prod_{i=1}^{n} n_i\right) &= \frac{1}{2^n}\{1-\sum_{i=1}^{n}\eta+\sum_{i<j}^{n}\mathrm{Pf}\left(A_{x_i,x_i+1,x_j,x_j+1}\right) \\
&\quad +\cdots+(-1)^n\,\mathrm{Pf}\left(A_{x_1,x_1+1,\ldots,x_n,x_n+1}\right)\}.
\end{aligned}$$

By lemma 5.1.3 we have

$$\begin{aligned}
\mathrm{E}_{t\to\infty}\left(\prod_{i=1}^{n} n_i\right) &= \frac{1}{2^n}\{1-\sum_{i=1}^{n}\eta+\sum_{i<j}^{n}\eta^2+\cdots+(-1)^n\,\eta^n\} \\
&= \frac{1}{2^n}\{1-n\eta+\binom{n}{2}\eta^2+\cdots+(-1)^n\,\eta^n\} \\
&= \left[\frac{1-\eta}{2}\right]^n.
\end{aligned}$$

So the steady state of annihilating random walk is Bernoulli.

□

5.2 Spontaneous creation of particles in CRW

Although we can prove that ARW with immigration preserves the Pfaffian property, the counterpart in CRW is not obvious.

In this section we will show that why CRW with spontaneous creation of particles does not act like the counterpart of ARW and thus is unlikely to be a Pfaffian point process.

We will derive the kinetic equation of empty interval probability for CRW with immigration and that of the Pfaffian of empty interval probabilities of single intervals. By observing the two kinetic equations are different we can thus conclude that CRW with immigration may not be a Pfaffian point process.

Lemma 35. *In the case of CRW with immigration, the empty interval probability* $P_t\left(\Omega_{x_1,y_1}\cap\cdots\cap\Omega_{x_n,y_n}\right) = P_t\left(\underline{x},\underline{y}\right)$ *satisfies the following kinetic equation:*

$$\partial_t P_t\left(\underline{x},\underline{y}\right) = D_e\Delta P_t\left(\underline{x},\underline{y}\right) - C\left[\sum_{i=1}^{n}|y_i-x_i|\right]P_t\left(\underline{x},\underline{y}\right).$$

Proof The discussion from (3.2) to (3.6) is still valid but now we have to also consider the effect of the spontaneous creation of particles. Also here we denote the rate of hopping of the particles by D_e. Now we assume that the probability that there will be a spontaneous creation of a particle in an empty interval $[x_i, y_i]$ in δt is proportional to the width of the interval $y_i - x_i$, i.e.

$$C\left|y_i - x_i\right| P\left(\Omega_{x_i,y_i}\right)$$

where C is the rate of spontaneous creation of particles. Therefore,

$$\begin{aligned} & P_{t+\delta t}\left(\underline{x},\underline{y}\right) - P_t\left(\underline{x},\underline{y}\right) \\ = \; & D_e\left(\delta t\right)\sum_{i=1}^{n}\left[\left(\partial_{x_i}^{-}\partial_{x_i}^{+} + \partial_{y_i}^{-}\partial_{y_i}^{+}\right)\right] P_t\left(\underline{x},\underline{y}\right) - \left[\sum_{i=1}^{n}\left|y_i - x_i\right|\right] C\left(\delta t\right) P_t\left(\underline{x},\underline{y}\right). \end{aligned}$$

Therefore,

$$\partial_t P_t\left(\underline{x},\underline{y}\right) = D_e \Delta P_t\left(\underline{x},\underline{y}\right) - C\left[\sum_{i=1}^{n}\left|y_i - x_i\right|\right] P_t\left(\underline{x},\underline{y}\right)$$

where $\Delta = \sum_{i=1}^{n}\left[\left(\partial_{x_i}^{-}\partial_{x_i}^{+} + \partial_{y_i}^{-}\partial_{y_i}^{+}\right)\right]$.

□

Lemma 36. *Empty interval probability for multiple intervals for CRW with immigration is not a Pfaffian of empty interval probability for single intervals.*

Proof Suppose empty interval probability $P_t\left(\Omega_{x_1,y_1} \cap \cdots \cap \Omega_{x_n,y_n}\right) = P_t\left(\underline{x},\underline{y}\right)$ is a Pfaffian $\mathrm{Pf}\left[P_t\left(z_i, z_j\right)\right]$ where z_i is x_i or y_i.

We derive the kinetic equation of this Pfaffian now. By the definition of Pfaffian we have

$$\begin{aligned} & \partial_t \mathrm{Pf}\left[P_t\left(\Omega_{z_i,z_j}\right)\right] \\ = \; & \partial_t\left\{\sum_{\sigma\in S_{2n}} \mathrm{sgn}\left(\sigma\right)\prod_{i=1}^{n} P_t\left(\Omega_{\sigma(2i-1),\sigma(2i)}\right)\right\} \end{aligned}$$

where we have put the constraint that $\sigma\left(2i-1\right) < \sigma\left(2i\right)$ and $\sigma\left(2i\right) < \sigma\left(2i+2\right)$.

Since

$$\partial_t P_t\left(\Omega_{\sigma(2i-1),\sigma(2i)}\right) = \left[D_e\left(\Delta_{\sigma(2i-1)} + \Delta_{\sigma(2i)}\right) - C\left|\sigma\left(2i\right) - \sigma\left(2i-1\right)\right|\right] P_t\left(\Omega_{\sigma(2i-1),\sigma(2i)}\right),$$

we get

$$
\begin{aligned}
& \partial_t \mathrm{Pf}\left[P_t\left(\Omega_{z_i,z_j}\right)\right] \\
= & \left\{\sum_{\sigma\in S_{2n}} \mathrm{sgn}\left(\sigma\right)\left[D_e\Delta - C\sum_{j=1}^{n}\left|\sigma\left(2i\right)-\sigma\left(2i-1\right)\right|\right]\prod_{i=1}^{n} P_t\left(\Omega_{\sigma(2i-1),\sigma(2i)}\right)\right\} \\
= & D_e\Delta \mathrm{Pf}\left[P_t\left(z_i,z_j\right)\right] - C\sum_{\sigma\in S_{2n}} \mathrm{sgn}\left(\sigma\right)\left[\sum_{j=1}^{n}\left|\sigma\left(2i\right)-\sigma\left(2i-1\right)\right|\right]\prod_{i=1}^{n} P_t\left(\Omega_{\sigma(2i-1),\sigma(2i)}\right).
\end{aligned}
$$

As the term $\sum_{j=1}^{n}\left|\sigma\left(2i\right)-\sigma\left(2i-1\right)\right|$ is different for every permutation σ, we cannot collect the terms and we do not have a partial differential equation for such a Pfaffian.

Therefore the Pfaffian does not satisfy the kinetic equation for the empty interval probability $P_t\left(\Omega_{x_1,y_1}\cap\cdots\cap\Omega_{x_n,y_n}\right)=P_t\left(\underline{x},\underline{y}\right)$ and therefore cannot be it.

$\square$

Since

$$
\rho_n^{CRW}\left(x_1,\ldots,x_n\right)=1-\sum_{i=1}^{n}P_t\left[\Omega_{x_i^-,x_i^+}\right]+\sum_{1\le i<j\le n}P_t\left[\Omega_{x_i^-,x_i^+,x_j^-,x_j^+}\right]+\cdots+(-1)^n P_t\left[\Omega_{x_1^-,x_1^+,\ldots,x_n^-,x_n^+}\right],
$$

but the empty interval probabilities are not Pfaffian, the sum of the summations cannot be combined to a single Pfaffian by Lemma 1. However, at this point we cannot prove that the correlation function for CRW with immigration of particles cannot be a Pfaffian of any functions.

5.3 Asymmetric ARW

In the previous discussion our transition rate of spins depends on both of the neighbouring spins and the effect of them are weighted equally. Neither the left nor the right neighbour has a larger effect over the other. This corresponds to symmetric ARW. In this section we want to relax this restriction and investigate the case of asymmetric ARW. Therefore we have to redefine our transition rate in Glauber model as

$$
\begin{aligned}
\omega\left(s_{k-1},s_k,s_{k+1}\right) &= 1+s_k\left(\gamma^{(-)}s_{k-1}+\gamma^{(+)}s_{k+1}\right) \\
&= 1+2\gamma s_k\left(\left(p\right)s_{k-1}+\left(1-p\right)s_{k+1}\right).
\end{aligned}
$$

where $\gamma^{(-)},\gamma^{(-)}\in[-1,0]$ and

$$
\gamma^{(-)}+\gamma^{(-)}=2\gamma.
$$

This is called the directed Ising model [13][14].

The parameters $\gamma^{(-)}$ and $\gamma^{(-)}$ represent the dependence of the spin s_k on its neighbours s_{k-1} and s_{k+1} respectively. For example, consider the spin configuration

$$|s_{k-1}s_ks_{k-1}\rangle = |++-\rangle.$$

The rate that s_k will flip from $+$ to $-$ is $1+2\left(\gamma^{(-)}-\gamma^{(+)}\right)$. let us consider two extreme cases. If $\gamma^{(-)}=2\gamma$ and $\gamma^{(+)}=0$, then the rate is $1+2\gamma$ and the spin s_k will have a small chance to flip since γ is negative. On the other hand, if $\gamma^{(-)}=0$ and $\gamma^{(+)}=2\gamma$, then the rate is $1-2\gamma$ and the spin s_k will have a bigger chance to flip from $+$ to $-$. In terms of domain wall, the first case means the ARW at k is not likely to move to the vacancy $k-1$ while the second case means the ARW at k is very likely to move to $k-1$. Therefore $\gamma^{(-)}$ and $\gamma^{(-)}$ represent how asymmetric ARW is . The more negative $\gamma^{(-)}$ is, the less likely an ARW will move to the negative direction. Similar argument goes for $\gamma^{(+)}$.

In the following section I will prove that under maximal entrance law asymmetric ARW still preserves the Pfaffian property by two approaches. Firstly let us consider the Fermionic approach.

5.3.1 Fermionic representation

$$\mathfrak{L}_{asym} = 2\sum_k \psi_k^\dagger\left(\psi_k+\gamma^{(-)}\left(\psi_{k-1}-\psi_{k-1}^\dagger\right)+\gamma^{(+)}\left(\psi_{k+1}+\psi_{k+1}^\dagger\right)\right)$$

Since $\mathfrak{L}_{asym}$ is quadratic in fermions ψ_k and $\psi_k^\dagger$, by Lemma 3 the new time-dependent fermions $\psi_k(t) = e^{\mathfrak{L}_{asym}t}\psi_k e^{-\mathfrak{L}_{asym}t}$ and $\psi_k^\dagger(t) = e^{\mathfrak{L}_{asym}t}\psi_k^\dagger e^{-\mathfrak{L}_{asym}t}$ can also be written as a summation of ψ_k and $\psi_k^\dagger$. Thus by Lemma 4 we can computer the anti-commutator $\left\{\psi_i^-(t),\psi_j(t)\right\} = \mathrm{E}_t(s_is_j)$ and therefore by Theorem 5 we will also have the Pfaffian property of spin correlation in this directed Ising model under the maximal entrance law as initial condition.

As again we have the Pfaffian property of spin correlation, the proof of Theorem 21 will be exactly the same and therefore asymmetric ARW preserves its Pfaffian property under the maximal entrance law.

5.3.2 Kinetic equation

By modifying the argument in Section 3.2 we can get

$$
\begin{aligned}
&\partial_t \mathrm{E}_t\left(s_{k_1} s_{k_2} \ldots s_{k_{2n}}\right) \\
= \;& -4nD\mathrm{E}_t\left(s_{k_1} s_{k_2} \ldots s_{k_{2n}}\right) - 2D\sum_{i=1}^{2n}\sum_{\vec{s}}\left[s_{k_i}\left(\gamma^{(-)} s_{k_i-1} + \gamma^{(+)} s_{k_i+1}\right)\right] P_t\left(\vec{s}\right)\left(s_{k_1} s_{k_2} \ldots s_{k_{2n}}\right) \\
= \;& -4nD\mathrm{E}_t\left(s_{k_1} s_{k_2} \ldots s_{k_{2n}}\right) - 2D\sum_{i=1}^{2n}\sum_{\vec{s}} P_t\left(\vec{s}\right)\left(s_{k_1} \ldots s_{k_{i-1}}\left(\gamma^{(-)} s_{k_i-1} + \gamma^{(+)} s_{k_i+1}\right) s_{k_{i+1}} \ldots s_{k_{2n}}\right) \\
= \;& D\sum_{i=1}^{2n}\sum_{\vec{s}} P_t\left(\vec{s}\right)\left(s_{k_1} \ldots s_{k_{i-1}}\left(-2\gamma^{(-)} s_{k_i-1} - 2s_{k_i} - 2\gamma^{(+)} s_{k_i+1}\right) s_{k_{i+1}} \ldots s_{k_{2n}}\right) \\
= \;& D\sum_{i=1}^{2n} \mathrm{E}_t\left(s_{k_1} \ldots s_{k_{i-1}}\left(-2\gamma^{(-)} s_{k_i-1} - 2s_{k_i} - 2\gamma^{(+)} s_{k_i+1}\right) s_{k_{i+1}} \ldots s_{k_{2n}}\right).
\end{aligned}
$$

Define $\partial_x^+ f(x) = f(x+1) - f(x)$, then we have

$$
\begin{aligned}
&\partial_t \mathrm{E}_t\left(s_{k_1} s_{k_2} \ldots s_{k_{2n}}\right) \\
= \;& D\sum_{i=1}^{2n} \mathrm{E}_t[s_{k_1} \ldots s_{k_{i-1}}(-2\gamma^{(+)}\left(s_{k_i-1} - 2s_{k_i} + s_{k_i+1}\right) - 2\left(\gamma^{(-)} - \gamma^{(+)}\right)\left(s_{k_i+1} - s_{k_i}\right) \\
& -2\left(\gamma^{(-)} + \gamma^{(+)} + 1\right) s_{k_i}) s_{k_{i+1}} \ldots s_{k_{2n}}] \\
= \;& \left[-2D\gamma^{(+)}\Delta - 2\left(\gamma^{(-)} - \gamma^{(+)}\right) D\sum_{i=1}^{2n}\partial_{x_i}^+ - 4nD\left(\gamma^{(-)} + \gamma^{(+)} + 1\right)\right]\mathrm{E}_t\left(s_{k_1} s_{k_2} \ldots s_{k_{2n}}\right).
\end{aligned}
$$

Thus we have a PDE of the form

$$
\partial_t u\left(\underline{x}\right) = \left[A\Delta - B + C\sum_{i=1}^{2n}\partial_{x_i}\right] u\left(\underline{x}\right)
$$

or

$$
\partial_t u\left(\underline{x}\right) = D\tilde{\Delta} u\left(\underline{x}\right) = D\sum_{i=1}^{2n}\left[a u\left(x_i + 1\right) - 2u\left(x_i\right) + b u\left(x_i - 1\right)\right] \tag{5.2}
$$

where $a, b \in [0,1]$ and $a + b = -4\gamma$.

By Lemma 16 we can see that the new kinetic equation generated by the new dynamics also has the form in Lemma 16 and therefore this new kinetic equation also preserves Pfaffian. Hence the spin correlation in the directed Ising model also has the Pfaffian structure.

Having this at our disposal we can go through the proof of Theorem 21 as before to prove the asymmetric ARW is a Pfaffian point process.

Notice that both proofs does not depend on the exactly temperature of the system, which means for any value of $\gamma \in [-1/2, 0]$ the proofs still hold. This means that asymmetric ARW with spontaneous creation of particles also preserves its Pfaffian property.

5.4 Asymmetric CRW

5.4.1 Kinetic equation of empty interval in asymmetric CRW

Although we cannot show that the general asymmetric CRW with immigration is a Pfaffian point process, in this section we are going to prove the special case that asymmetric CRW without immigration is also a Pfaffian point process.

We are going to calculate the kinetic equation of empty interval for asymmetric CRW and then show that it is the same as that of directed Ising model. By the uniqueness theorem in the following section we can then show that they are identical and use this result to prove asymmetric CRW is a Pfaffian point process.

Suppose we have a asymmetric CRW particle. Let the rate of hopping to the left be $2Dp$ and the rate of hopping to the right be $2D(1-p)$.

By imitating the equations (3.2), (3.3), (3.4) and (3.5), we get

$$[P_t(\Omega_{x_1,y_1} \cup \cdots \cup \Omega_{x_i+1,y_i} \cup \cdots \cup \Omega_{x_n,y_n}) - P_t(\Omega_{x_1,y_1} \cup \cdots \cup \Omega_{x_i,y_i} \cup \cdots \cup \Omega_{x_n,y_n})]\, 2D(p)(\delta t)$$

and

$$[P_t(\Omega_{x_1,y_1} \cup \cdots \cup \Omega_{x_i,y_i-1} \cup \cdots \cup \Omega_{x_n,y_n}) - P_t(\Omega_{x_1,y_1} \cup \cdots \cup \Omega_{x_i,y_i} \cup \cdots \cup \Omega_{x_n,y_n})]\, 2D(1-p)(\delta t)$$

which are the contribution from the interval (x_i, y_i) to the increase of the empty interval probability $P_t(\Omega_{x_1,y_1} \cap \cdots \cap \Omega_{x_n,y_n})$ in the time duration δt. Similarly for the decrease of the probability $P_t(\Omega_{x_1,y_1} \cap \cdots \cap \Omega_{x_n,y_n})$ we have

$$-[P_t(\Omega_{x_1,y_1} \cup \cdots \cup \Omega_{x_i,y_i} \cup \cdots \cup \Omega_{x_n,y_n}) - P_t(\Omega_{x_1,y_1} \cup \cdots \cup \Omega_{x_i-1,y_i} \cup \cdots \cup \Omega_{x_n,y_n})]\, 2D(1-p)(\delta t)$$

and

$$-[P_t(\Omega_{x_1,y_1} \cup \cdots \cup \Omega_{x,y} \cup \cdots \cup \Omega_{x_n,y_n}) - P_t(\Omega_{x_1,y_1} \cup \cdots \cup \Omega_{x_i,y_i+1} \cup \cdots \cup \Omega_{x_n,y_n})]\, 2D(p)(\delta t).$$

Summing up the contributions from all the intervals (x_i, y_i) we have

$$\partial_t P_t\left(\underline{x}, \underline{y}\right) = D\tilde{\Delta} P_t\left(\underline{x}, \underline{y}\right)$$

where $\tilde{\Delta} = \sum_{i=1}^{n}\left(\tilde{\Delta}_{x_i} + \tilde{\Delta}_{y_i}\right)$ and

$$\tilde{\Delta}_{x_i} f\left(x_i\right) = 2pf\left(x_i + 1\right) - 2f\left(x_i\right) + 2\left(1 - p\right) f\left(x_i - 1\right)$$

which is the same as (5.2) if we set $a = 2p$ and $b = 2\left(1 - p\right)$.

Since only the dynamics of the system is changed the initial conditions and the boundary conditions remain the same as the symmetric case.

Now we only have to prove the uniqueness of the kinetic equation to show that the empty interval probability is identical to the spin correlation.

5.4.2 Uniqueness of $\partial_t u = D\tilde{\Delta} u$

We would like to obtain a uniqueness theorem for the PDE (5.2).

We will imitate the development in Section 5.1.2 to develop the uniqueness theorem for this PDE.

Lemma 37. *For a function $u : \Omega \times [0, \infty) \to R$, the following*

$$\begin{cases} D\tilde{\Delta} u \leq u_t & in\ \Omega \times [0, \infty) \\ u \geq 0 & on\ \partial\Omega \times [0, \infty) \cup \Omega \times \{0\} \end{cases}$$

implies

$$u \geq 0\ in\ \bar{\Omega} \times [0, \infty)$$

where D is a positive real number, $\tilde{\Delta}$ is the discrete Laplacian operator defined in (5.2)*, u_t is the derivative of u with respect to time and $\Omega \subset Z^n$ is a bounded domain.*

Proof This lemma mirrors Lemma 27. Firstly, assume that u attains minimum on $D \times [0, T]$ at (x^*, t^*). This can be found since Ω is bounded and u is continuous with respect to time and there are finite number of spatial grid points in Ω.

Now suppose $u\left(x^*, t^*\right) < 0$, otherwise the proof is done.

And

$$
\begin{aligned}
& \tilde{\Delta} u\left(x^*, t^*\right) \\
\geq\ & D \sum_{i=1}^n\left[-a\left|u\left(x_i+1\right)\right|-2 u\left(x_i\right)-b\left|u\left(x_i-1\right)\right|\right] \\
\geq\ & D \sum_{i=1}^n\left[-\left|u\left(x_i+1\right)\right|-2 u\left(x_i\right)-\left|u\left(x_i-1\right)\right|\right] \\
\geq\ & 0
\end{aligned}
$$

as (x^*, t^*) is the minimum point. But by assumption $u_t(x^*, t^*) \leq 0$ and thus this leads to the contradiction to the assumption that $D\tilde{\Delta}u \leq u_t$. Therefore $u \geq 0$ in $\bar{\Omega} \times [0, \infty)$.

□

Lemma 38. *For a bounded function $u : \Omega \times [0, \infty) \to R$, the following*

$$
\begin{cases} D\tilde{\Delta} - u_t \leq -\delta < 0 & \text{in } \Omega \times [0,\infty) \\ u \geq 0 & \text{on } \partial\Omega \times [0,\infty) \cup \Omega \times \{0\} \end{cases}
$$

implies

$$u \geq 0 \text{ in } \bar{\Omega} \times [0, \infty)$$

where D is a positive real number, $\tilde{\Delta}$ is the discrete Laplacian operator defined in (5.2), *u_t is the derivative of u with respect to time, $\delta > 0$ and $\Omega \subset Z^n$ is an unbounded domain.*

Proof This lemma mirrors Lemma 29. Define

$$v = u + \epsilon|x|^2.$$

Then

$$D\tilde{\Delta}v - v_t = D\tilde{\Delta}u - u_t + \epsilon D\tilde{\Delta}|x|^2.$$

where

$$\tilde{\Delta}|x|^2 = n\left(-4\gamma - 2\right)|x|^2 + 2\left(a - b\right)\sum_{i=1}^n x_i + \left(-4\gamma\right)n.$$

If we choose a diameter R big enough such that

$$\epsilon \leq \frac{\delta}{D\left(2R + 2n\right)}$$

then $D\tilde{\Delta}v - v_t \leq 0$. Then by Lemma 37 we have $v \geq 0$ inside $|x|^2 \leq R^2$.

We can choose R so big that the term $\epsilon|x|^2$ will dominate and thus $v \geq 0$ for $|x|^2 \geq R^2$.

In summary, $v \geq 0$ in $\bar{\Omega} \times [0,\infty)$ and hence $u \geq 0$ in $\bar{\Omega} \times [0,\infty)$ as ϵ is arbitrary. □

Lemma 39. *For a bounded function $u : \Omega \times [0,\infty) \to R$, the following*

$$\begin{cases} D\tilde{\Delta}u - u_t \leq 0 & \text{in } \Omega \times [0,\infty) \\ u \geq 0 & \text{on } \partial\Omega \times [0,\infty) \cup \Omega \times \{0\} \end{cases}$$

implies

$$u \geq 0 \text{ in } \bar{\Omega} \times [0,\infty)$$

where D is a positive real number, $\tilde{\Delta}$ is the discrete Laplacian operator defined in (5.2), *u_t is the derivative of u with respect to time, $\delta > 0$ and $\Omega \subset Z^n$ is an unbounded domain.*

Proof This lemma mirrors Lemma 29 and the proof is almost exactly the same. Define

$$v = u + \delta t.$$

It can be seen that

$$D\tilde{\Delta}v - v_t = D\tilde{\Delta}u - u_t - \delta\,(1 + Bt) < 0.$$

By lemma 38 we know that

$$v \geq 0 \text{ in } \bar{\Omega} \times [0,\infty)$$

and since δ can be arbitrarily small we have

$$u \geq 0 \text{ in } \bar{\Omega} \times [0,\infty).$$

□

Lemma 40. *For a bounded function $u : \Omega \times [0,\infty) \to R$, the following*

$$\begin{cases} D\tilde{\Delta}u - u_t \geq 0 & \text{in } \Omega \times [0,\infty) \\ u \leq 0 & \text{on } \partial\Omega \times [0,\infty) \cup \Omega \times \{0\} \end{cases}$$

implies

$$u \leq 0 \text{ in } \bar{\Omega} \times [0,\infty)$$

where D is a positive real number, $\tilde{\Delta}$ is the discrete Laplacian operator defined in (5.2), *u_t is the derivative of u with respect to time, $\delta > 0$ and $\Omega \subset Z^n$ is an unbounded domain.*

Proof This lemma mirrors Lemma 30.

Substitute $v = -u$ into lemma 39 then we can immediately get the result.

□

Theorem 41. *If a bounded function $u : \Omega \times [0, \infty) \to R$ satisfies the kinetic equation*

$$\partial_t u = D\tilde{\Delta} u \text{ in } \Omega \times [0, \infty)$$

and a boundary condition

$$u = f \text{ on } \partial\Omega \times [0, \infty) \cup \Omega \times \{0\}$$

then u is unique.

Proof This theorem mirrors Theorem 31 and the proof is almost exactly the same. Suppose there are two bounded functions u and v which satistify heat equation

$$\Delta u = u_t \text{ in } \Omega \times [0, \infty)$$

and

$$\Delta v = v_t \text{ in } \Omega \times [0, \infty)$$

and the same boundary condition

$$u = v = f \text{ on } \partial\Omega \times [0, \infty) \cup \Omega \times \{0\}.$$

Then let $w = u - v$ and we can see

$$\Delta w = w_t \text{ in } \Omega \times [0, \infty)$$

and

$$w = 0 \text{ on } \partial\Omega \times [0, \infty) \cup \Omega \times \{0\}.$$

By lemma 39 and 40 we have

$$w = 0 \text{ in } \bar{\Omega} \times [0, \infty).$$

Therefore we have proved the uniqueness of u. □

Therefore by Theorem 41, Theorem 23 still holds for asymmetric case and can be used to prove Theorem 24 for asymmetric case. So asymmetric CRW is also a Pfaffian point process.

5.5 Position-dependent random walk

In the previous sections we have generalised the ARW and CRW by introducing immigration and asymmetry. To further generalise the models, in this section we will investigate ARW and CRW which have different bias at different position .

5.5.1 ARW

As before, we can investigate ARW via Glauber model. Once we can prove the spin correlation in the new system is still a Pfaffian then we can prove the new ARW is also a Pfaffian point process by the argument in Theorem 21. By generalising the flipping rate to

$$\omega\left(s_{k-1}, s_k, s_{k+1}\right) = 1 + s_k\left(\gamma_k^{(-)} s_{k-1} + \gamma_k^{(+)} s_{k+1}\right)$$

we can thus study whether this more general ARW is still a Pfaffian point process.

As before, we can study the problem in two approaches. Firstly we will observe the change in the Liouville operator $\mathfrak{L}$. The operator becomes

$$\mathfrak{L} = 2\sum_k \psi_k^\dagger\left(\psi_k + \gamma_k^{(-)}\left(\psi_{k-1} - \psi_{k-1}^\dagger\right) + \gamma_k^{(+)}\left(\psi_{k+1} + \psi_{k+1}^\dagger\right)\right).$$

Thus it is still quadratic in fermions and therefore the spin correlation in the system is still a Pfaffian. Therefore our more general ARW is still a Pfaffian point process by the argument similar to those in the previous sections.

Next we can study the problem by observing the kinetic equation of the spin correlation. By an argument similar to Lemma 26 we get

$$\begin{aligned}
&\partial_t \mathrm{E}_t\left(s_{k_1} s_{k_2} \ldots s_{k_{2n}}\right) \\
= \;& D\sum_{i=1}^{2n} \mathrm{E}_t\left(s_{k_1}\ldots s_{k_{i-1}}\left(-2\gamma_{k_i}^{(-)} s_{k_i-1} - 2s_{k_i} - 2\gamma_{k_i}^{(+)} s_{k_i+1}\right) s_{k_{i+1}}\ldots s_{k_{2n}}\right) \qquad (5.3) \\
= \;& D\sum_{i=1}^{2n} \mathrm{E}_t[s_{k_1}\ldots s_{k_{i-1}}(-2\gamma_{k_i}^{(+)}\left(s_{k_i-1} - 2s_{k_i} + s_{k_i+1}\right) - 2\left(\gamma_{k_i}^{(-)} - \gamma_{k_i}^{(+)}\right)\left(s_{k_i+1} - s_{k_i}\right) \\
& -2\left(\gamma_{k_i}^{(-)} + \gamma_{k_i}^{(+)} + 1\right) s_{k_i}) s_{k_{i+1}}\ldots s_{k_{2n}}] \\
= \;& \left\{-2D\sum_{i=1}^{2n}\left[\gamma_{k_i}^{(+)}\Delta_{k_i} + \left(\gamma_{k_i}^{(-)} - \gamma_{k_i}^{(+)}\right)\partial_{x_i}^{+} + \left(\gamma_{k_i}^{(-)} + \gamma_{k_i}^{(+)} + 1\right)\right]\right\}\mathrm{E}_t\left(s_{k_1} s_{k_2}\ldots s_{k_{2n}}\right),
\end{aligned}$$

which is a partial differential equation of the form in Lemma 16.

So the new dynamics of the system will preserve the Pfaffian property of the spin correlation and therefore the new ARW is also a Pfaffian point process.

5.5.2 CRW

CRW is a bit less general but we can limit our interest to the case in which there is no immigration of particles. To prove CRW with position-dependent bias is a Pfaffian point process it suffices to show that the empty interval probability in this new system is also a Pfaffian as this is the only change in the proof of Theorem 24.

Let the particle at position x_i have the hopping rate to the left $2Dp_{x_i}$ and the hopping rate to the right $2D\left(1-p_{x_i}\right)$.

By imitating the argument in Section 5.4.1 we get the kinetic equation for the empty interval probability in the new system

$$\partial_t P_t\left(\underline{x}, \underline{y}\right) = D\tilde{\Delta} P_t\left(\underline{x}, \underline{y}\right)$$

where $\tilde{\Delta} = \sum_{i=1}^{n}\left(\tilde{\Delta}_{x_i} + \tilde{\Delta}_{y_i}\right)$ and

$$\tilde{\Delta}_{x_i} f\left(x_i\right) = 2p_{x_i} f\left(x_i+1\right) - 2f\left(x_i\right) + 2\left(1-p_{x_i}\right) f\left(x_i-1\right).$$

which is the same partial differential equation as equation (5.3) in the special case that $\gamma_k = \frac{1}{2}\left(\gamma_k^{(-)} + \gamma_k^{(+)}\right) = \frac{-1}{2}$ for all k.

Therefore our new empty interval probability in this more general system is also a Pfaffian and thus CRW is a Pfaffian point process in this new system.

5.6 One-sided initial condition

5.6.1 ARW

In chapter 4 we proved that ARW and CRW are Pfaffian point processes under maximal entrance law. In this section we will investigate the case in which the initial condition is one-sided. In the case of Glauber model this means

$$|s_k\rangle = \begin{cases} |\uparrow\rangle & \text{for } k \leq 0 \\ \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right) & \text{for } k > 0. \end{cases}$$

We will consider the most general random walk we have discussed so far, which is the position dependent random walk in Section 5.5.

Lemma 42. *Under the one-sided initial condition and at zero temperature the spin correlation $E_t(s_{x_1}\cdots s_{x_{2n}})$ is a Pfaffian.*

Proof Here we use the approach described in section 3.4.

The only thing we have to prove is that the one-sided initial condition is a Pfaffian. After this has been proved, by the uniqueness of discrete kinetic equation (5.3) and Theorem 15 in section 3.3, the spin correlation is a Pfaffian.

Let $x_i < x_j$ for $i < j$.

Case i: all the spins are on the left hand side of the origin, i.e. $k \leq 0$ for all $k = 1, \dots, 2n$.

Since the spin correlation is

$$\mathrm{E}_{t=0}(s_{x_1}\cdots s_{x_{2n}}) = \mathrm{E}_{t=0}(s_{x_1})\cdots \mathrm{E}_{t=0}(s_{x_{2n}}) = 1$$

and S is

$$S_{i,j} = (-1)^{\chi(i>j)}\mathrm{E}_{t=0}(s_i s_j) = (-1)^{\chi(i>j)}1$$

and hence

$$\mathrm{Pf}(S) = 1.$$

So the initial conditions agree.

Case ii: all the spins are on the right hand side of the origin, i.e. $k > 0$ for all $k = 1, \dots, 2n$.

The spin correlation is

$$\mathrm{E}_{t=0}(s_{x_1}\cdots s_{x_{2n}}) = \mathrm{E}_{t=0}(s_{x_1})\cdots \mathrm{E}_{t=0}(s_{x_{2n}}) = 0$$

and S is

$$S_{i,j} = (-1)^{\chi(i>j)}\mathrm{E}_{t=0}(s_i s_j) = 0$$

and hence $\mathrm{Pf}(S) = 0$ and therefore the initial conditions agree.

Case iiii: some spins are on the right hand side of the origin, i.e. $k > 0$ for at least one $k \in \{1, \dots, 2n\}$.

Suppose $x_{2n} > 0$, then $\mathrm{E}_{t=0}(s_{x_{2n}}) = 0$.Therefore

$$\mathrm{E}_{t=0}(s_{x_1}\cdots s_{x_{2n}}) = \mathrm{E}_{t=0}(s_{x_1})\cdots \mathrm{E}_{t=0}(s_{x_{2n}}) = 0$$

and since the $2n$-th row of the matrix S will be zero as $\mathrm{E}_{t=0}(s_{x_k}s_{x_{2n}}) = \mathrm{E}_{t=0}(s_{x_k})\,\mathrm{E}_{t=0}(s_{x_{2n}}) = 0$ and therefore $\mathrm{Pf}(S) = 0$. So the initial conditions agree.

Therefore by the uniquess of discrete heat equation the spin correlation is also a Pfaffian under this initial condition.

□

With this lemma now we can prove that ARW under this new initial condition is also a Pfaffian point process at zero temperature.

Theorem 43. *For an initial condition that every site on the right hand side of the origin has independent $\frac{1}{2}$ probability being occupied and the left hand side of the origin being empty, the correlation function of ARW at zero temperature is Pfaffian:*

$$\rho_n^{ARW}(x_1, \ldots, x_n; t) = \frac{Pf(I-S)}{2^n} = \frac{Pf(K)}{2^n}$$

where I, S and K are the same as defined in Theorem 21 .

Proof The proof goes through nearly the same as that of Theorem 21. The only difference is that we use Lemma 42 instead of Theorem 5 to change the Pfaffians $\mathrm{Pf}(S|_{J_2})$ to spin correlations. The rest is just the same.

□

5.6.2 CRW

We will only prove that CRW without immigration preserves its Pfaffian property for the new one-sided initial condition. The one-sided initial condition for CRW is a bit different from that of ARW because of the thinning relation [16]. To imitate Theorem 9 we can show that:

Lemma 44. *The probability $P_t[\Omega_{x_1,y_1} \cap \cdots \cap \Omega_{x_n,y_n}]$ and the spin correlation function $E(s_{x_1}s_{y_1}\cdots s_{x_n}s_{y_n})$ are identical equations by the uniqueness theorem of the heat equation, i.e.*

$$P_t(\Omega_{x_1,y_1} \cap \cdots \cap \Omega_{x_n,y_n}) = E_t(s_{x_1}s_{y_n}\cdots s_{x_n}s_{y_n})$$

if given the initial condition for the CRW:

$$P_{t=0}(x_k \text{ is occupied}) = \begin{cases} 1 & \text{for } k > 0 \\ 0 & \text{for } k \leq 0 \end{cases}$$

and the initial condition for the Glauber model:

$$|s_k\rangle = \begin{cases} |\uparrow\rangle & \text{for } k \leq 0 \\ \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) & \text{for } k > 0 \end{cases}$$

where $x_1 < y_1 < x_2 < y_2 < \cdots < x_n < y_n$.

Proof In Section 5.5.2 we have already shown that both functions satisfy the same discrete kinetic equation and in Theorem 9 the same boundary condition. If we can show that their initial conditions agree, then by uniqueness of discrete kinetic equation they are identical functions.

In the proof of lemma 42 we know that $\mathrm{E}_{t=0}\left(s_{x_1}s_{y_n}\cdots s_{x_n}s_{y_n}\right) = 0$ if any $x_k > 0$ or $y_k > 0$; otherwise $\mathrm{E}_{t=0}\left(s_{x_1}s_{y_n}\cdots s_{x_n}s_{y_n}\right) = 1$ if all $x_k \leq 0$ and $y_k \leq 0$.

From the definition of empty interval probability we can see that

$$P_t\left[\Omega_{x_1,y_1} \cap \cdots \cap \Omega_{x_n,y_n}\right] = \begin{cases} 0 & \text{if } x_k, y_k < 0\ \forall k \\ 1 & \text{otherwise.} \end{cases}$$

Therefore the initial conditions agree and hence by the uniqueness of discrete heat equation the functions are identical. □

With this lemma we are ready to prove the generalisation of Theorem 24.

Theorem 45. *For an initial condition that*

$$P_{t=0}\left(x_k \textit{ is occupied}\right) = \begin{cases} 1 & \textit{for } k > 0 \\ 0 & \textit{for } k \leq 0, \end{cases}$$

the correlation function of CRW is Pfaffian:

$$\rho_n^{CRW}\left(x_1, \ldots, x_n; t\right) = 2^n Pf(I - S) = 2^n Pf(K)$$

where the definition of the matrices I, S *and* K *are the same as before.*

Proof Instead of using Theorem 23 we use lemma 44. The rest of the proof is just the same as that of Theorem 24. □

Chapter 6

Multi-time ARW as a Pfaffian Point Process

We are interested in proving that the multi-time correlation function of ARW is also a Pfaffian point process, i.e. we want to show

$$\mathrm{E}\left[\prod_{i=1}^{n} n_{t_i,z_i}\right] = \mathrm{Pf}\left[K\right]$$

where K is a $2n \times 2n$ anti-symmetric matrix which has a kernel of

$$K\left(x,y\right) = \begin{pmatrix} K_{1,1}\left(x,y\right) & K_{1,2}\left(x,y\right) \\ K_{2,1}\left(x,y\right) & K_{2,2}\left(x,y\right) \end{pmatrix} \tag{6.1}$$

and also satisfies the identity

$$K_{i,j}\left(x,y\right) = -K_{j,i}\left(y,x\right), \tag{6.2}$$

which is just another way of stating that K is an anti-symmetric matrix.

However, instead of proving it directly like we did in the previous chapters, we prove it by considering a more general ARW-spin mixed correlation and by showing that it is also a Pfaffian point process, i.e.

$$\mathrm{E}\left[\prod_{i=1}^{n} n_{t_i,z_i} \prod_{j=1}^{2m} s_{t,y_j}\right] = \mathrm{Pf}\left[K\right].$$

where K has the kernel of (6.1) and satisfies the identity (6.2).

We will firstly investigate the multi-time spin correlation to show that it is a Pfaffian of a matrix. Then we will consider the special case in which all the spins in the earlier time slots are paired up and show that it is a Pfaffian point process. Using this result as a jumping board we will show that the ARW-spin correlation is a Pfaffian point process as ARW can be viewed as domain walls between pairs of spins.

6.1 Multi-time spin correlation

6.1.1 The kinetic equation

PDE for general multi-time spin correlation

We start with only two spins at two different time slots and investigate the multi-time spin correlation.

Two-point function

Suppose we have two spins s_{x_1,t_1} and s_{x_2,t_2}. Let

$$C_2\left(x_1,t_1;x_2,t_2\right)=\mathrm{E}\left[s_{x_1,t_1}s_{x_2,t_2}\right].$$

Lemma 46.

$$\partial_{t_2}C_2\left(x_1,t_1;x_2,t_2\right)=\left[\left(-2\gamma D\right)\Delta_2+\left(-2D\right)\left(1+2\gamma\right)\right]C_2\left(x_1,t_1;x_2,t_2\right)$$

where $\Delta_2=\partial_{x_2}^2$.

Proof Since

$$C_2\left(x_1,t_1;x_2,t_2\right)=\sum_{\vec{s},\vec{s'}}P\left[\vec{s_{t_1}}=\vec{s'},\vec{s_{t_2}}=\vec{s}\right]s'_{x_1}s_{x_2},$$

where $\vec{s}$ and $\vec{s'}$ are spin configurations and s_{x_1} and s'_{x_2} are the values of the spins at x_1 and x_2 in spin configurations $\vec{s}$ and $\vec{s'}$ respectively. We have

$$\partial_{t_2}C_2\left(x_1,t_1;x_2,t_2\right)=\sum_{\vec{s},\vec{s'}}\partial_{t_2}P\left[\vec{s_{t_1}}=\vec{s'},\vec{s_{t_2}}=\vec{s}\right]s'_{x_1}s_{x_2}.$$

For the general case, we have

$$
\begin{aligned}
\partial_{t_2} P\left[\vec{s_{t_1}}=\vec{s'}, \vec{s_{t_2}}=\vec{s}\right] &= -\sum_{\vec{s_1}} w\left(\vec{s}, \vec{s_1}\right) P\left[\vec{s_{t_1}}=\vec{s'}, \vec{s_{t_2}}=\vec{s}\right] \\
&\quad +\sum_{\vec{s_2}} w\left(\vec{s_2}, \vec{s}\right) P\left[\vec{s_{t_1}}=\vec{s'}, \vec{s_{t_2}}=\vec{s_2}\right]
\end{aligned}
$$

where $w(\vec{s}, \vec{s_1})$ and $w(\vec{s_2}, \vec{s})$ are the rate of the spin configuration transiting from $\vec{s}$ to $\vec{s_1}$ and from $\vec{s_2}$ to $\vec{s}$ respectively. The rate of transition w is independent of time t because the system is assumed to be a stationary Markov chain, which is a reasonable assumption for a physical system.

The definition of $w\left(\vec{s}, \vec{s'}\right)$ for $\vec{s} \neq \vec{s'}$ is

$$
w\left(\vec{s}, \vec{s'}\right)=\lim_{\delta t \rightarrow 0} \frac{P\left(\vec{s_{t+\delta t}}=\vec{s'} | \vec{s_t}=\vec{s}\right)}{\delta t}.
$$

For $\vec{s}=\vec{s'}$,

$$
\begin{aligned}
w(\vec{s}, \vec{s}) &= \lim_{\delta t \rightarrow 0} \frac{P\left(\vec{s_{t+\delta t}}=\vec{s'} | \vec{s_t}=\vec{s}\right)-1}{\delta t} \\
&= -\sum_{\vec{\tilde{\sigma}} \neq \vec{s}} w\left(\vec{s}, \vec{\tilde{s}}\right).
\end{aligned}
$$

So in the following we only have to consider the case in which $\vec{s} \neq \vec{s'}$.

In the case of Glauber model, we have

$$
w\left(\vec{s}, \vec{s'}\right)=\begin{cases} D\left[1+\gamma s_k\left(s_{k-1}+s_{k+1}\right)\right] & \text{if } \vec{s} \text{ and } \vec{s'} \text{ only differ at one site } k \\ 0 & \text{if } \vec{s} \text{ and } \vec{s'} \text{ differ at more than one site.} \end{cases}
$$

where D is a positive constant and $\gamma=\tanh\left(\frac{2J}{kT}\right)$.

So now we have

$$
\begin{aligned}
\partial_{t_2} P\left[\vec{s_{t_1}}=\vec{s}, \vec{s_{t_2}}=\vec{s'}\right] &= -D \sum_{x \in \mathrm{Z}}\left[1+\gamma s_x\left(s_{x-1}+s_{x+1}\right)\right] P\left[\vec{s_{t_1}}=\vec{s'}, \vec{s_{t_2}}=\vec{s}\right] \\
&\quad +D \sum_{x \in \mathrm{Z}}\left[1-\gamma s_x\left(s_{x-1}+s_{x+1}\right)\right] F_x\left[\vec{s_{t_1}}=\vec{s'}, \vec{s_{t_2}}=\vec{s}\right]
\end{aligned}
$$

where $D\left[1+\gamma s_x\left(s_{x-1}+s_{x+1}\right)\right]$ is the flipping rate of the spin s_x at position x

and $F_x\left[\vec{s_{t_1}}=\vec{s'},\vec{s_{t_2}}=\vec{s}\right]$ is the probability that $\vec{s_{t_1}}=\vec{s'}$ and $\vec{s_{t_2}}=\vec{s_x}$, where $\vec{s_x}$ differs from $\vec{s}$ only at x.

Therefore,

$$\begin{aligned}
&\partial_{t_2}C_2\left(x_1,t_1;x_2,t_2\right)\\
=\ & D\sum_{x\in\mathbb{Z}}\sum_{\vec{s},\vec{s'}}\left\{-P\left[\vec{s_{t_1}}=\vec{s'},\vec{s_{t_2}}=\vec{s}\right]+F_x\left[\vec{s_{t_1}}=\vec{s'},\vec{s_{t_2}}=\vec{s}\right]\right\}s'_{x_1}s_{x_2}\\
&+\left\{P\left[\vec{s_{t_1}}=\vec{s'},\vec{s_{t_2}}=\vec{s}\right]+F_x\left[\vec{s_{t_1}}=\vec{s'},\vec{s_{t_2}}=\vec{s}\right]\right\}\left[-\gamma s_x\left(s_{x-1}+s_{x+1}\right)\right]s'_{x_1}s_{x_2}
\end{aligned}\tag{6.3}$$

Firstly consider the first term in (6.3),

$$D\sum_{x\in\mathbb{Z}}\sum_{\vec{s},\vec{s'}}\left\{-P\left[\vec{s_{t_1}}=\vec{s'},\vec{s_{t_2}}=\vec{s}\right]+F_x\left[\vec{s_{t_1}}=\vec{s'},\vec{s_{t_2}}=\vec{s}\right]\right\}s'_{x_1}s_{x_2}.\tag{6.4}$$

For $x\in\{x_2\}$,

$$\sum_{\vec{s},\vec{s'}}\left\{-P\left[\vec{s_{t_1}}=\vec{s'},\vec{s_{t_2}}=\vec{s}\right]\right\}s'_{x_1}s_{x_2}=\sum_{\vec{s},\vec{s'}}\left\{F_x\left[\vec{s_{t_1}}=\vec{s'},\vec{s_{t_2}}=\vec{s}\right]\right\}s'_{x_1}s_{x_2}.$$

For $x\notin\{x_2\}$,

$$\sum_{\vec{s},\vec{s'}}\left\{P\left[\vec{s_{t_1}}=\vec{s'},\vec{s_{t_2}}=\vec{s}\right]\right\}s'_{x_1}s_{x_2}=\sum_{\vec{s},\vec{s'}}\left\{F_x\left[\vec{s_{t_1}}=\vec{s'},\vec{s_{t_2}}=\vec{s}\right]\right\}s'_{x_1}s_{x_2}.$$

Hence (6.4) is equal to

$$-2D\sum_{\vec{s},\vec{s'}}\left\{P\left[\vec{s_{t_1}}=\vec{s'},\vec{s_{t_2}}=\vec{s}\right]\right\}s'_{x_1}s_{x_2}.\tag{6.5}$$

Now consider the second term in (6.3),

$$D\sum_{x\in\mathbb{Z}}\sum_{\vec{s},\vec{s'}}\left\{P\left[\vec{s_{t_1}}=\vec{s'},\vec{s_{t_2}}=\vec{s}\right]+F_x\left[\vec{s_{t_1}}=\vec{s'},\vec{s_{t_2}}=\vec{s}\right]\right\}\left[-\gamma s_x\left(s_{x-1}+s_{x+1}\right)\right]s'_{x_1}s_{x_2}\tag{6.6}$$

For $x\in\{x_2\}$,

$$\left[-\gamma s_{x_2}\left(s_{x_2-1}+s_{x_2+1}\right)\right]s'_{x_1}s_{x_2}=\left[-\gamma\left(s_{x_2-1}+s_{x_2+1}\right)\right]s'_{x_1}$$

since $s_x^2 = 1$ for all x and therefore the term is independent of x_2 and hence

$$\begin{aligned}
& \sum_{\vec{s},\vec{s'}} \left\{ P\left[\vec{s_{t_1}} = \vec{s'}, \vec{s_{t_2}} = \vec{s} \right] \right\} \left[-\gamma s_{x_2} \left(s_{x_2-1} + s_{x_2+1} \right) \right] s'_{x_1} s_{x_2} \\
= & \sum_{\vec{s},\vec{s'}} \left\{ F_{x_2} \left[\vec{s_{t_1}} = \vec{s'}, \vec{s_{t_2}} = \vec{s} \right] \right\} \left[-\gamma s_{x_2} \left(s_{x_2-1} + s_{x_2+1} \right) \right] s'_{x_1} s_{x_2}.
\end{aligned}$$

For $x \notin \{x_2\}$,

$$\begin{aligned}
& \sum_{\vec{s},\vec{s'}} \left\{ P\left[\vec{s_{t_1}} = \vec{s'}, \vec{s_{t_2}} = \vec{s} \right] \right\} \left[-\gamma s_{x_2} \left(s_{x_2-1} + s_{x_2+1} \right) \right] s'_{x_1} s_{x_2} \\
= & \sum_{\vec{s},\vec{s'}} \left\{ -F_{x} \left[\vec{s_{t_1}} = \vec{s'}, \vec{s_{t_2}} = \vec{s} \right] \right\} \left[-\gamma s_{x_2} \left(s_{x_2-1} + s_{x_2+1} \right) \right] s'_{x_1} s_{x_2}.
\end{aligned}$$

Therefore, (6.6) is equal to

$$2D \sum_{\vec{s},\vec{s'}} \left\{ P\left[\vec{s_{t_1}} = \vec{s'}, \vec{s_{t_2}} = \vec{s} \right] \right\} \left[\gamma s_{x_2} \left(s_{x_2-1} + s_{x_2+1} \right) \right] s'_{x_1} s_{x_2}. \tag{6.7}$$

Summarising the results (6.5) and (6.7) we get (6.3) is equal to

$$\begin{aligned}
& -2D \sum_{\vec{s},\vec{s'}} \left\{ P\left[\vec{s_{t_1}} = \vec{s'}, \vec{s_{t_2}} = \vec{s} \right] \right\} \left[1 + \gamma s_{x_2} \left(s_{x_2-1} + s_{x_2+1} \right) \right] s'_{x_1} s_{x_2} \\
= & -2D \sum_{\vec{s},\vec{s'}} \left\{ P\left[\vec{s_{t_1}} = \vec{s'}, \vec{s_{t_2}} = \vec{s} \right] \right\} \left[s_{x_2} + \gamma \left(s_{x_2-1} + s_{x_2+1} \right) \right] s'_{x_1} \\
= & -2D \sum_{\vec{s},\vec{s'}} \left\{ P\left[\vec{s_{t_1}} = \vec{s'}, \vec{s_{t_2}} = \vec{s} \right] \right\} \gamma \left[-2 s_{x_2} + \left(s_{x_2-1} + s_{x_2+1} \right) \right] s'_{x_1} \\
& + \left\{ P\left[\vec{s_{t_1}} = \vec{s'}, \vec{s_{t_2}} = \vec{s} \right] \right\} \left(1 + 2\gamma \right) s_{x_2} s'_{x_1} \\
= & \left[\left(-2\gamma D \right) \Delta_2 + \left(-2D \right) \left(1 + 2\gamma \right) \right] C_2 \left(x_1, t_1; x_2, t_2 \right).
\end{aligned}$$

So the lemma is proved. □

Notice that when $\gamma = -\frac{1}{2}$ it becomes

$$\partial_{t_2} C_2 \left(x_1, t_1; x_2, t_2 \right) = D \Delta_2 C_2 \left(x_1, t_1; x_2, t_2 \right).$$

which is exactly the discrete heat equation.

$2n$ spins

We proceed to the case of general $2n$ spins in different time slots. Assume that we have already known that the multi-time spin correlation of $2n$ spins $s_{x_1}, \ldots, s_{x_{2n}}$ at k times satisfies the kinetic equation. To show that this is also true in the case of $k+1$ times, we proceed in two steps.

Firstly we will consider the case in which $2n-1$ spins $s_{x_1}, \ldots, s_{x_{2n-1}}$ are in k times $t_1, \ldots, t_k$ and 1 spin $s_{x_{2n}}$ at t_{k+1}.

Lemma 47.

$$\partial_{t_{k+1}} C_{2n}\left(x_1, \ldots, t_1; \ldots; x_{2n}, t_{k+1}\right) = \left[(-2\gamma D)\, \Delta_{2n} + (-2D)\,(1+2\gamma)\right] C_{2n}\left(x_1, \ldots, t_1; \ldots; x_{2n}, t_{k+1}\right).$$

Proof Since

$$C_{2n}\left(x_1, \ldots, t_1; \ldots; x_{2n}, t_{k+1}\right) = \sum_{\vec{s}^{(1)}, \ldots, \vec{s}^{(k+1)}} P\left[\vec{s_{t_1}} = \vec{s}^{(1)}, \ldots, \vec{s_{t_{k+1}}} = \vec{s}^{(k+1)}\right] \left(s_{x_1}^{(1)} \cdots s_{x_{2n}}^{(k+1)}\right),$$

we have

$$\begin{aligned}\partial_{t_{k+1}} C_{2n} &= D \sum_{x \in \mathbb{Z}} \sum_{\vec{s}^{(1)}, \ldots, \vec{s}^{(k+1)}} \left[-P + F_x\right] \left(s_{x_1}^{(1)} \cdots s_{x_{2n}}^{(k+1)}\right) \\ &\quad + \left[P + F_x\right] \left[-\gamma s_x^{(k+1)} \left(s_{x-1}^{(k+1)} + s_{x+1}^{(k+1)}\right)\right] \left(s_{x_1}^{(1)} \cdots s_{x_{2n}}^{(k+1)}\right)\end{aligned}$$

where $C_{2n} = C_{2n}\left(x_1, \ldots, t_1; \ldots; x_{2n}, t_{k+1}\right)$, $P = P\left[\vec{s_{t_1}} = \vec{s}^{(1)}, \ldots, \vec{s_{t_{k+1}}} = \vec{s}^{(k+1)}\right]$ and $F_x = P\left[\vec{s_{t_1}} = \vec{s}^{(1)}, \ldots, \vec{s_{t_{k+1}}} = \vec{s}_x^{(k+1)}\right]$, where $\vec{s}_x^{(k+1)}$ differs from $\vec{s}^{(k+1)}$ only at x.

Similar to the proof of the previous lemma, we have

$$\sum_{x \in \mathbb{Z}} \sum_{\vec{s}^{(1)}, \ldots, \vec{s}^{(k+1)}} \left[-P + F_x\right] \left(s_{x_1}^{(1)} \cdots s_{x_{2n}}^{(k+1)}\right) = \sum_{\vec{s}^{(1)}, \ldots, \vec{s}^{(k+1)}} \left[-2P\right] \left(s_{x_1}^{(1)} \cdots s_{x_{2n}}^{(k+1)}\right)$$

and

$$\begin{aligned}&\sum_{x \in \mathbb{Z}} \sum_{\vec{s}^{(1)}, \ldots, \vec{s}^{(k+1)}} \left[P + F_x\right] \left[-\gamma s_x^{(k+1)} \left(s_{x-1}^{(k+1)} + s_{x+1}^{(k+1)}\right)\right] \left(s_{x_1}^{(1)} \cdots s_{x_{2n}}^{(k+1)}\right) \\ = &\sum_{\vec{s}^{(1)}, \ldots, \vec{s}^{(k+1)}} \left[2P\right] \left[-\gamma s_{x_{2n}}^{(k+1)} \left(s_{x_{2n}-1}^{(k+1)} + s_{x_{2n}+1}^{(k+1)}\right)\right] \left(s_{x_1}^{(1)} \cdots s_{x_{2n}}^{(k+1)}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}
\partial_{t_{k+1}} C_{2n} &= -2D \sum_{\vec{s}^{(1)},\dots,\vec{s}^{(k+1)}} [P] \left(s_{x_1}^{(1)} \cdots s_{x_{2n}}^{(k+1)} \right) \left[1 + \gamma s_{x_{2n}}^{(k+1)} \left(s_{x_{2n}-1}^{(k+1)} + s_{x_{2n}+1}^{(k+1)} \right) \right] \\
&= \left[(-2\gamma D)\, \Delta_{2n} + (-2D)(1+2\gamma) \right] C_{2n}
\end{aligned}$$

□

Now we will consider the case in which $2n-m$ spins $s_{x_1}, \dots, s_{x_{2n-m}}$ are in k times $t_1, \dots, t_k$ and m spins $s_{x_{2n-m+1}}, \dots, s_{x_{2n}}$ at t_{k+1}.

Lemma 48.

$$\begin{aligned}
& \partial_{t_{k+1}} C_{2n} \left(x_1, \dots, t_1; \dots; x_{2n-m+1}, \dots, x_{2n}, t_{k+1} \right) \\
= \; & \left[(-2\gamma D)\, \Delta_{2n-m+1,\dots,2n} + m\,(-2D)(1+2\gamma) \right] C_{2n} \left(x_1, \dots, t_1; \dots; x_{2n-m+1}, \dots, x_{2n}, t_{k+1} \right).
\end{aligned}$$

Proof As befre, we have

$$\begin{aligned}
\partial_{t_{k+1}} C_{2n} = \; & D \sum_{x\in \mathbb{Z}} \sum_{\vec{s}^{(1)},\dots,\vec{s}^{(k+1)}} \left[-P + F_x \right] \left(s_{x_1}^{(1)} \cdots s_{x_{2n}}^{(k+1)} \right) \\
& + \left[P + F_x \right] \left[-\gamma s_x^{(k+1)} \left(s_{x-1}^{(k+1)} + s_{x+1}^{(k+1)} \right) \right] \left(s_{x_1}^{(1)} \cdots s_{x_{2n}}^{(k+1)} \right)
\end{aligned}$$

where $C_{2n} = C_{2n}\left(x_1, \dots, t_1; \dots; x_{2n-m+1}, \dots, x_{2n}, t_{k+1} \right)$, $P = P\left[\vec{s_{t_1}} = \vec{s}^{(1)}, \dots, \vec{s_{t_{k+1}}} = \vec{s}^{(k+1)} \right]$ and $F_x = P\left[\vec{s_{t_1}} = \vec{s}^{(1)}, \dots, \vec{s_{t_{k+1}}} = \vec{s}_x^{(k+1)} \right]$, where $\vec{s}_x^{(k+1)}$ differs from $\vec{s}^{(k+1)}$ only at x.

Since for $x \in \{x_{2n-m+1}, \dots, x_{2n}\}$,

$$\sum_{\vec{s}^{(1)},\dots,\vec{s}^{(k+1)}} \left[-P + F_x \right] \left(s_{x_1}^{(1)} \cdots s_{x_{2n}}^{(k+1)} \right) = \sum_{\vec{s}^{(1)},\dots,\vec{s}^{(k+1)}} \left[-2P \right] \left(s_{x_1}^{(1)} \cdots s_{x_{2n}}^{(k+1)} \right)$$

and for $x \notin \{x_{2n-m+1}, \dots, x_{2n}\}$,

$$\sum_{\vec{s}^{(1)},\dots,\vec{s}^{(k+1)}} \left[-P + F_x \right] \left(s_{x_1}^{(1)} \cdots s_{x_{2n}}^{(k+1)} \right) = 0,$$

so we have

$$\sum_{x\in \mathbb{Z}} \sum_{\vec{s}^{(1)},\dots,\vec{s}^{(k+1)}} \left[-P + F_x \right] \left(s_{x_1}^{(1)} \cdots s_{x_{2n}}^{(k+1)} \right) = m \sum_{\vec{s}^{(1)},\dots,\vec{s}^{(k+1)}} \left[-2P \right] \left(s_{x_1}^{(1)} \cdots s_{x_{2n}}^{(k+1)} \right).$$

Since for $x \in \{x_{2n-m+1}, \ldots, x_{2n}\}$,

$$\begin{aligned}
&\sum_{\vec{s}^{(1)},\ldots,\vec{s}^{(k+1)}} [P] \left[-\gamma s_x^{(k+1)} \left(s_{x-1}^{(k+1)} + s_{x+1}^{(k+1)}\right)\right] \left(s_{x_1}^{(1)} \cdots s_{x_{2n}}^{(k+1)}\right) \\
= &\sum_{\vec{s}^{(1)},\ldots,\vec{s}^{(k+1)}} [F_x] \left[-\gamma s_x^{(k+1)} \left(s_{x-1}^{(k+1)} + s_{x+1}^{(k+1)}\right)\right] \left(s_{x_1}^{(1)} \cdots s_{x_{2n}}^{(k+1)}\right)
\end{aligned}$$

and for $x \notin \{x_{2n-m+1}, \ldots, x_{2n}\}$,

$$\begin{aligned}
&\sum_{\vec{s}^{(1)},\ldots,\vec{s}^{(k+1)}} [P] \left[-\gamma s_x^{(k+1)} \left(s_{x-1}^{(k+1)} + s_{x+1}^{(k+1)}\right)\right] \left(s_{x_1}^{(1)} \cdots s_{x_{2n}}^{(k+1)}\right) \\
= &\sum_{\vec{s}^{(1)},\ldots,\vec{s}^{(k+1)}} [-F_x] \left[-\gamma s_x^{(k+1)} \left(s_{x-1}^{(k+1)} + s_{x+1}^{(k+1)}\right)\right] \left(s_{x_1}^{(1)} \cdots s_{x_{2n}}^{(k+1)}\right),
\end{aligned}$$

so we have

$$\begin{aligned}
&D \sum_{x\in\mathbb{Z}} \sum_{\vec{s}^{(1)},\ldots,\vec{s}^{(k+1)}} [P + F_x] \left[-\gamma s_x^{(k+1)} \left(s_{x-1}^{(k+1)} + s_{x+1}^{(k+1)}\right)\right] \left(s_{x_1}^{(1)} \cdots s_{x_{2n}}^{(k+1)}\right) \\
= \; &D \sum_{i=1}^{m} \sum_{\vec{s}^{(1)},\ldots,\vec{s}^{(k+1)}} [2P] \left[-\gamma s_{x_{2n-m+i}}^{(k+1)} \left(s_{x_{2n-m+i}-1}^{(k+1)} + s_{x_{2n-m+i}+1}^{(k+1)}\right)\right] \left(s_{x_1}^{(1)} \cdots s_{x_{2n}}^{(k+1)}\right).
\end{aligned}$$

Combining the terms we have

$$\begin{aligned}
&-2D \sum_{i=1}^{m} \sum_{\vec{s}^{(1)},\ldots,\vec{s}^{(k+1)}} [P] \left[1 + \gamma s_{x_{2n-m+i}}^{(k+1)} \left(s_{x_{2n-m+i}-1}^{(k+1)} + s_{x_{2n-m+i}+1}^{(k+1)}\right)\right] \left(s_{x_1}^{(1)} \cdots s_{x_{2n}}^{(k+1)}\right) \\
= \; &\left[(-2\gamma D)\, \Delta_{2n-m+1,\ldots,2n} + m\,(-2D)\,(1+2\gamma)\right] C_{2n}.
\end{aligned}$$

$\square$

Kinetic equation of the multi-time Pfaffian Pf $\left[A\left(t_{x_i}, x_i; t_{x_j}, x_j\right)\right]$

Suppose there are m spins $s_{x_{2n-m+1}}, \ldots, s_{x_{2n}}$ at time t_{k+1} and $t_{x_i}, x_i; t_{x_j}, x_j$ is the multi-time Pfaffian which will be defined below. We want to prove a generalisation of Lemma 16 to multi-time case.

Lemma 49. *Suppose A is a $2n \times 2n$ anti-symmetric matrix whose entries $c_{i,j}$ are functions of positions x_i, x_j and the times t_{x_i}, t_{x_j}, i.e.*

$$a_{i,j} = (-1)^{\chi(i<j)}\, g\left(t_{x_i}, x_i; t_{x_j}, x_j\right).$$

Let there be k times slots $t_1 < t_2 < \cdots < t_{k-1} < t$.

If

$$\begin{aligned}\partial_t a_{i,j} &= \left[b_i\partial_i^2 + b_j\partial_j^2 + c_i\partial_i + c_j\partial_j + \tilde{f}\left(x_i\right) + \tilde{f}\left(x_j\right)\right] a_{i,j} \\ &= \left[\sum_{i=l}^{2n} b_l\partial_l^2 + \sum_{m=1}^{2n} c_m\partial_m + \tilde{f}\left(x_i\right) + \tilde{f}\left(x_j\right)\right] a_{i,j}\end{aligned}$$

where

$$\tilde{f}\left(x_i\right) = \begin{cases} f\left(x_i\right) & \textit{if } t_{x_i} = t \\ 0 & \textit{otherwise} \end{cases},$$

and $\partial_l = \frac{\partial}{\partial_{x_l}}$, b_l and c_m are functions of x_i and $f\left(\underline{x}\right)$ is a function of $x_1, \ldots, x_{2n}$, then

$$\partial_t Pf(A) = \left[\sum_{l=1}^{2n} b_l\partial_l^2 + c_l\partial_l + \tilde{f}\left(x_l\right)\right] Pf(A)\,.$$

Proof By the definition of Pfaffian it can be expressed as

$$\operatorname{Pf}\left(A\right) = \frac{1}{2^n n!}\sum_{\sigma\in S_{2n}} \operatorname{sgn}\left(\sigma\right)\prod_{i=1}^{n} a_{\sigma(2i-1),\sigma(2i)}$$

where σ is the symmetric group and $\mathrm{sgn}(\sigma)$ is the signature of σ. Therefore,

$$
\begin{aligned}
\partial_t \mathrm{Pf}(A) &= \frac{1}{2^n n!}\sum_{\sigma\in S_{2n}} \mathrm{sgn}(\sigma) \\
&\quad \sum_{j=0}^{n-1}\left[\prod_{i=1}^{j} a_{\sigma(2i-1),\sigma(2i)}\left(\partial_t a_{\sigma(2j-1),\sigma(2j)}\right)\prod_{i=j+2}^{n} a_{\sigma(2i-1),\sigma(2i)}\right] \\
&= \frac{1}{2^n n!}\sum_{\sigma\in S_{2n}} \mathrm{sgn}(\sigma) \\
&\quad \sum_{j=0}^{n-1}\{\prod_{i=1}^{j} a_{\sigma(2i-1),\sigma(2i)} \\
&\quad [b_{\sigma(2i-1)}\partial^2_{\sigma(2i-1)} + b_{\sigma(2i)}\partial^2_{\sigma(2i)} + \\
&\quad c_{\sigma(2i-1)}\partial_{\sigma(2i-1)} + c_{\sigma(2i)}\partial_{\sigma(2i)} + \tilde{f}\left(x_{\sigma(2j-1)}\right) + \tilde{f}\left(x_{\sigma(2j)}\right)]a_{\sigma(2j-1),\sigma(2j)} \\
&\quad \prod_{i=j+2}^{n} a_{\sigma(2i-1),\sigma(2i)}\} \\
&= \frac{1}{2^n n!}\sum_{\sigma\in S_{2n}} \mathrm{sgn}(\sigma)\left[\sum_{l=1}^{2n} b_l\partial_l^2 + c_l\partial_l + \tilde{f}(x_l)\right]\left(\prod_{i=1}^{n} a_{\sigma(2i-1),\sigma(2i)}\right) \\
&= \left[\sum_{l=1}^{2n} b_l\partial_l^2 + c_l\partial_l + \tilde{f}(x_l)\right]\mathrm{Pf}(A)
\end{aligned}
$$

□

This lemma will be used in the following sections to prove that the various Pfaffians that we will see later satisfy the same kinetic equation as the various correlations.

6.1.2 Single-time paired spin correlation

Although we have already proved the single-time spin correlation in Theorem 5, we only did it for a specific ordering. Now suppose we have a paired spin correlation of this form

$$
\mathrm{E}_t\left[\prod_{i=1}^{N} s_{z_i}s_{z_i^+}\prod_{j=1}^{2m} s_{y_j}\right].
$$

We know it equals to the Pfaffian of the matrix stated in Theorem 5 for the ordering $z_1 < z_1^+ \le z_2 < \cdots \le z_N < z_N^+ \le y_1 \le \cdots \le y_{2m}$. We want to expand our interest of ordering to consider a specific set of orderings such that no spins s_{y_j} will be between any pairs of spins s_{z_i} and $s_{z_i^+}$ and we still hold the constraints

$$y_1 < y_2 < \cdots < y_{2m}$$

and

$$z_1 < z_1^+ < z_2 < \cdots < z_{2m} < z_{2m}^+.$$

Our purpose is to find such a matrix that its Pfaffian is the paired spin correlation. Firstly we prove that this Pfaffian satisfies the same kinetic equation as before.

Lemma 50.

$$\partial_t Pf\left[K_s^1\right] = \left[(-2\gamma D)\,\Delta_{1,\dots,2n+2m} + (2n+2m)\,(-2D)\,(1+2\gamma)\right] Pf\left[K_s^1\right] \tag{6.8}$$

where $\Delta_{1,\dots,2n+2m} = \sum_{j=1}^{m} \partial_{x_{i_j}^2}$ *is the discrete Laplace differential operator with respect to the* $2n+2m$ *variables* x_i, ξ_i *and* y_j. *And* K_s^1 *is the matrix corresponding to the single-time paired spin correlation whose blocks are*

$$K_s^1\left(z_i,\xi_i;z_i,\xi_i\right) = \begin{pmatrix} 0 & c_t(z_i,\xi_i) \\ -c_t(z_i,\xi_i) & 0 \end{pmatrix}$$

which are on the diagonal of the first $2n \times 2n$ *rows and columns, and*

$$K_s^1\left(z_i,\xi_i;z_j,\xi_j\right) = \begin{pmatrix} c_t\left(z_i,z_j\right) & c_{t_1}\left(z_i,\xi_j\right) \\ c_t\left(\xi_i,z_j\right) & c_{t_1}\left(\xi_i,\xi_j\right) \end{pmatrix}$$

above the diagonal of the first $2n \times 2n$ *rows and columns. The* 2×1 *blocks are in the first* $2n$ *rows and last* $2m$ *columns. They are:*

$$K_s^1\left(z_i,\xi_i;y_j\right) = \begin{pmatrix} \tilde{c}_t\left(z_i,y_j\right) \\ c_t\left(\xi_i,y_j\right) \end{pmatrix}.$$

where $c_t(x,y)$ *is the single-time spin correlation which satisfies the kinetic equation*

$$\partial_t c_t(x,y) = \left[(-2\gamma D)\,\Delta_{x,y} + (-2D)\,(1+2\gamma)\right] c_t(x,y)$$

which is proved in Lemma 46 and has the boundary condition,

$$c_t(x,y) = \begin{cases} E_t(s_x s_y) & \textit{if } x < y \\ -E_t(s_x s_y) & \textit{if } x > y \\ 1 & \textit{otherwise} \end{cases},$$

and $\tilde{c}_t(x,y)$*is similar except that it has a different boundary condition,*

$$\tilde{c}_t(x,y) = \begin{cases} E_t(s_x s_y) & \textit{if } x < y \\ -E_t(s_x s_y) & \textit{if } x > y \\ -1 & \textit{otherwise} \end{cases}.$$

The 1×1 *blocks are in the last* $2m$ *rows and last* $2m$ *columns. They are:*

$$K_s^1(y_i; y_j) = c_t(y_i, y_j).$$

Proof Let $-b_l = 2\gamma D$ for all $l = 2n - m + 1, \ldots, 2n$, $c_m = 0$ for all m and $f(\underline{x}) = (-2D)(1 + 2\gamma)$. Both $c_t(x,y)$ and $\tilde{c}_t(x,y)$ satisfy the same kinetic equation

$$\partial_t c_t(x,y) = \left[\sum_{i=l}^{2n} b_l \partial_l^2 + \sum_{m=1}^{2n} c_m \partial_m + \tilde{f}_{i,j}(\underline{x})\right] c_t(x,y),$$

therefore all the entries satisfy the same kinetic equation and thus by Lemma 16 the theorem is proved.

□

Now we want to show that the Pfaffian satisfies the same boundary condition as the paired spin correlation. The set of ordering of interest is that

$$y_1 < y_2 < \cdots < y_{2m}$$

and

$$z_1 < \xi_1 < z_2 < \cdots < z_{2m} < \xi_{2m}.$$

and no s_{y_j} is between any pairs of spins s_{z_i} and s_{ξ_i}. Since the Pfaffian we are considering has entries of different functions instead of only one function, we cannot directly use Lemma 18. However, the boundary condition is still preserved. We will prove it in two steps. Firstly we will prove that on the boundary the Pfaffian is reduced to a certain form.

Lemma 51. *When $y_j = y_{j+1}$, $z_i = \xi_i$, $\xi_i = z_{i+1}$ or $\xi_j = y_i$ the Pfaffian $Pf\left[K_s^1\right]$ becomes*

$$
Pf\begin{pmatrix} 0 & 1 & B \\ -1 & 0 & B \\ -B^T & -B^T & A \end{pmatrix}
$$

where B is a $1 \times (2n+2m-2)$ row matrix and A is an anti-symmetric $(2n+2m-2) \times (2n+2m-2)$ matrix obtained from A by removing the rows and columns corresponding to the boundary conditions.

Proof There are 4 cases to consider:

1. $y_j = y_{j+1}$
2. $z_i = \xi_i$
3. $\xi_i = z_{i+1}$
4. $\xi_j = y_i$.

For $y_j = y_{j+1}$ the columns corresponding to y_j and y_{j+1} are

$$
\begin{array}{cccc}
\cdots & \tilde{c}_t\left(z_1, y_i\right) & \tilde{c}_t\left(z_1, y_{i+1}\right) & \cdots \\
\cdots & c_t\left(\xi_1, y_i\right) & c_t\left(\xi_1, y_{i+1}\right) & \cdots \\
 & \vdots & \vdots & \\
\cdots & \tilde{c}_t\left(z_n, y_i\right) & \tilde{c}_t\left(z_n, y_{i+1}\right) & \cdots \\
\cdots & c_t\left(\xi_n, y_i\right) & c_t\left(\xi_n, y_{i+1}\right) & \cdots \\
\cdots & c_t\left(y_1, y_i\right) & c_t\left(y_1, y_{i+1}\right) & \cdots \\
 & \vdots & \vdots & \\
\cdots & c_t\left(y_{i-1}, y_i\right) & c_t\left(y_{i-1}, y_{i+1}\right) & \cdots \\
\cdots & \mathbf{0} & \mathbf{c_t\left(y_i, y_{i+1}\right)} & \cdots \\
\cdots & -\mathbf{c_t\left(y_i, y_{i+1}\right)} & \mathbf{0} & \cdots \\
\cdots & -c_t\left(y_i, y_{i+2}\right) & -c_t\left(y_{i+1}, y_{i+2}\right) & \cdots \\
 & \vdots & \vdots & \\
\cdots & -c_t\left(y_i, y_{2m}\right) & -c_t\left(y_{i+1}, y_{2m}\right) & \cdots
\end{array}.
$$

For $z_i = \xi_i$ the columns corresponding to z_i and ξ_i are

$$\begin{array}{cccc}
\cdots & c_t\left(z_1, z_i\right) & c_t\left(z_1, \xi_i\right) & \cdots \\
\cdots & c_t\left(\xi_1, z_i\right) & c_t\left(\xi_1, \xi_1\right) & \cdots \\
 & \vdots & \vdots & \\
\cdots & \mathbf{0} & \mathbf{c_t}\left(\mathbf{z_i}, \boldsymbol{\xi}_\mathbf{i}\right) & \cdots \\
\cdots & -\mathbf{c_t}\left(\mathbf{z_i}, \boldsymbol{\xi}_\mathbf{i}\right) & \mathbf{0} & \cdots \\
 & \vdots & \vdots & \\
\cdots & -c_t\left(z_i, z_n\right) & -c_t\left(\xi_i, z_n\right) & \cdots \\
\cdots & -c_t\left(z_i, \xi_n\right) & -c_t\left(\xi_i, \xi_n\right) & \cdots \\
\cdots & -\tilde{c}_t\left(z_i, y_1\right) & -c_t\left(\xi_i, y_1\right) & \cdots \\
 & \vdots & \vdots & \\
\cdots & -\tilde{c}_t\left(z_i, y_{2m}\right) & -c_t\left(\xi_i, y_{2m}\right) & \cdots
\end{array}.$$

For $\xi_i = z_{i+1}$ the columns corresponding to ξ_i and z_{i+1} are

$$\begin{array}{cccc}
\cdots & c_t\left(z_1, \xi_i\right) & c_t\left(z_1, z_{i+1}\right) & \cdots \\
\cdots & c_t\left(\xi_i, \xi_i\right) & c_t\left(\xi_1, z_{i+1}\right) & \cdots \\
 & \vdots & \vdots & \\
\cdots & c_t\left(z_i, \xi_i\right) & c_t\left(z_i, z_{i+1}\right) & \cdots \\
\cdots & \mathbf{0} & \mathbf{c_t}\left(\boldsymbol{\xi}_\mathbf{i}, \mathbf{z_{i+1}}\right) & \cdots \\
\cdots & -\mathbf{c_t}\left(\boldsymbol{\xi}_\mathbf{i}, \mathbf{z_{i+1}}\right) & \mathbf{0} & \cdots \\
 & \vdots & \vdots & \\
\cdots & -c_t\left(\xi_i, z_n\right) & -c_t\left(z_{i+1}, z_n\right) & \cdots \\
\cdots & -c_t\left(\xi_i, \xi_n\right) & -c_t\left(z_{i+1}, \xi_n\right) & \cdots \\
\cdots & -c_t\left(\xi_i, y_1\right) & -\tilde{c}_t\left(z_{i+1}, y_1\right) & \cdots \\
 & \vdots & \vdots & \\
\cdots & -c_t\left(\xi_i, y_{2m}\right) & -\tilde{c}_t\left(z_{i+1}, y_{2m}\right) & \cdots
\end{array}.$$

For $\xi_j = y_i$ the columns corresponding to ξ_i and y_j are

$$\begin{array}{ccccc}
\cdots & c_{t_1}(z_1, z_j) & \cdots & \tilde{c}_{t_1}(z_1, y_i) & \cdots \\
\cdots & c_{t_1}(\xi_1, z_j) & \cdots & c_{t_1}(\xi_1, y_i) & \cdots \\
 & \vdots & & \vdots & \\
\cdots & \mathbf{0} & \cdots & \mathbf{\tilde{c}_{t_1}(z_j, y_i)} & \cdots \\
\cdots & -c_{t_1}(z_j, \xi_j) & \cdots & c_{t_1}(\xi_j, y_i) & \cdots \\
 & \vdots & & \vdots & \\
\cdots & -c_{t_1}(z_j, z_n) & \cdots & \tilde{c}_{t_1}(z_n, y_i) & \cdots \\
\cdots & -c_{t_1}(z_j, \xi_n) & \cdots & c_{t_1}(\xi_n, y_i) & \cdots \\
\cdots & -\tilde{c}_{t_1}(z_j, y_1) & \cdots & c_{t_1}(y_1, y_i) & \cdots \\
 & \vdots & & \vdots & \\
\cdots & \mathbf{-\tilde{c}_{t_1}(z_j, y_i)} & \cdots & \mathbf{0} & \cdots \\
\cdots & -\tilde{c}_{t_1}(z_j, y_{i+1}) & \cdots & -c_{t_1}(y_i, y_{i+1}) & \cdots \\
 & \vdots & & \vdots & \\
\cdots & -\tilde{c}_{t_1}(z_j, y_{2m}) & \cdots & -c_{t_1}(y_i, y_{2m}) & \cdots
\end{array}.$$

In all the first 3 cases, the two columns will be identical except for the entries in bold font which will become

$$\begin{array}{cccc}
 & \vdots & \vdots & \\
\cdots & 0 & 1 & \cdots \\
\cdots & -1 & 0 & \cdots \\
 & \vdots & \vdots &
\end{array},$$

and in the last case $\xi_j = y_i$ the columns will become

$$\begin{array}{ccccc}
\cdots & 0 & \cdots & -1 & \cdots \\
 & \vdots & & \vdots & \\
\cdots & 1 & \cdots & 0 & \cdots
\end{array}.$$

The same goes for the corresponding rows. Therefore by even row and column permutations we can obtain the matrix stated in the lemma. □

Secondly we show that the Pfaffian of the matrix of the aforementioned form can be reduced to a Pfaffian of less order.

Lemma 52. *The Pfaffian of an $2n \times 2n$ anti-symmetric matrix of this form*

$$\begin{pmatrix} 0 & 1 & B \\ -1 & 0 & B \\ -B^T & -B^T & A \end{pmatrix}$$

is equal to $Pf[A]$, where B is a $1 \times (2n-2)$ row matrix and A is a $(2n-2) \times (2n-2)$ anti-symmetric matrix.

Proof By the Pfaffian identity

$$\mathrm{Pf}\,(A) = \sum_{i=2}^{2n} (-1)^i \, a_{1,i} \mathrm{Pf}\left(A_{\hat{1},\hat{i}}\right) \tag{6.9}$$

where A is a $2n \times 2n$ anti-symmetric matrix and $A_{\hat{1},\hat{i}}$ is a $(2n-2)\times(2n-2)$ anti-symmetric matrix obtained from A by removing the first and i-th row and column.

Then we have

$$\begin{aligned} & \mathrm{Pf}\begin{pmatrix} 0 & 1 & B \\ -1 & 0 & B \\ -B^T & -B^T & A \end{pmatrix} \\ = \quad & \mathrm{Pf}\,(A) + \sum_{i=3}^{2n} (-1)^i \, b_{i-2} \mathrm{Pf}\begin{pmatrix} 0 & B \\ -B^T & A \end{pmatrix}_{\widehat{i-1}} \end{aligned} \tag{6.10}$$

where $\begin{pmatrix} 0 & B \\ -B^T & A \end{pmatrix}_{\widehat{i-1}}$ is the matrix obtained from $\begin{pmatrix} 0 & B \\ -B^T & A \end{pmatrix}_{\widehat{i-1}}$ by removing the $i-1$-th column and row.

Then we apply (6.9) to $\mathrm{Pf}\begin{pmatrix} 0 & B \\ -B^T & A \end{pmatrix}_{\widehat{i-1}}$ again and obtain

$$\begin{aligned} & \mathrm{Pf}\begin{pmatrix} 0 & B \\ -B^T & A \end{pmatrix}_{\widehat{i-1}} \\ = \quad & \sum_{l=2}^{i-2} (-1)^l \, b_{l-1} \mathrm{Pf}\left(A_{\widehat{l-1},\widehat{i-2}}\right) + \sum_{k=i}^{2n-1} (-1)^{k-1} \, b_{k-1} \mathrm{Pf}\left(A_{\widehat{k-1},\widehat{i-2}}\right) \end{aligned}$$

Putting it back into (6.10) we have

$$\begin{aligned}
&\mathrm{Pf}\begin{pmatrix} 0 & 1 & B \\ -1 & 0 & B \\ -B^T & -B^T & A \end{pmatrix} \\
= \quad & \mathrm{Pf}\left(A\right) + \sum_{i=3}^{2n}\sum_{l=2}^{i-2} \left(-1\right)^{i+l} b_{i-2}b_{l-1}\mathrm{Pf}\left(A_{\hat{l-1},\hat{i-2}}\right) + \sum_{i=3}^{2n}\sum_{k=i}^{2n-1} \left(-1\right)^{i+k-1} b_{i-2}b_{k-1}\mathrm{Pf}\left(A_{\hat{k-1},\hat{i-2}}\right).
\end{aligned}$$

Let us define $b_{i-2}b_{l-1}\mathrm{Pf}\left(A_{\hat{l-1},\hat{i-2}}\right) = d_{i,l}$. Then the term

$$\sum_{i=3}^{2n}\sum_{l=2}^{i-2} \left(-1\right)^{i+l} b_{i-2}b_{l-1}\mathrm{Pf}\left(A_{\hat{l-1},\hat{i-2}}\right) + \sum_{i=3}^{2n}\sum_{k=i}^{2n-1} \left(-1\right)^{i+k-1} b_{i-2}b_{k-1}\mathrm{Pf}\left(A_{\hat{k-1},\hat{i-2}}\right)$$

is actually equal to the sum of these entries:

$$\begin{matrix}
0 & -d_{1,2} & d_{1,3} & -d_{1,4} & \cdots & -d_{1,2n-2} \\
d_{2,1} & 0 & -d_{2,3} & d_{2,4} & \cdots & d_{2,2n-2} \\
-d_{3,1} & d_{3,2} & 0 & -d_{3,4} & \cdots & -d_{3,2n-2}. \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
d_{2n-2,1} & -d_{2n-2,2} & d_{2n-2,3} & \cdots & d_{2n-2,2n-3} & 0
\end{matrix}$$

Therefore it is zero and thus the lemma is proved. □

Now we can prove that the single-time paired spin correlation is equal to the aforementioned Pfaffian Pf $\left[K_s^1\right]$.

Theorem 53.

$$E_t\left[\prod_{i=1}^{N} s_{z_i}s_{\xi_i}\prod_{j=1}^{2m} s_{y_j}\right] = Pf\left[K_s^1\right]$$

Proof By Lemma 48 and Lemma 50, both sides satisfy the same kinetic equation as single-time spin correlation is a special case of multi-time spin correlation.

By Lemma 51 and Lemma 52 and the property that $s^2 = 1$, both side satisfy the same boundary condition.

Therefore, under the maximum entrance law as the initial condition and by the uniqueness Theorem 31 they are identical.

□

If we let $\xi_i = z_i^+$ for all $i = 1, \ldots, n$ then we have

$$\mathrm{E}_t\left[\prod_{i=1}^{N} s_{z_i} s_{z_i^+} \prod_{j=1}^{2m} s_{y_j}\right] = \mathrm{Pf}\left[K_s^1\right],$$

which is special case of Theorem 53.

6.2 Multi-time ARW-spin correlation

6.2.1 Single-time ARW-spin correlation

In this section we will investigate the single-time mixed ARW-spin correlation. Using this as a steping stone we can see the natural generalisation of the result to multi-time case.

Theorem 54. *Under the maximal entrance law as initial condition,*

$$E\left[\prod_{i=1}^{N} n_{z_i} \prod_{j=1}^{2m} s_{y_j}\right] = \left(\frac{-1}{2}\right)^N Pf\left[K^1_{ARW-s}\left(z_i, z_j\right)\right]$$

where the i, j-th 2×2 block over the diagonal is

$$K^1_{ARW-s}\left(z_i, z_j\right) = \begin{pmatrix} c_t\left(z_i, z_j\right) & c_t\left(z_i, z_j^+\right) - c_t\left(z_i, z_j\right) \\ c_t\left(z_i^+, z_j\right) - c_t\left(z_i, z_j\right) & 2c_t\left(z_i, z_j\right) - c_t\left(z_i^+, z_j\right) - c_t\left(z_i, z_j^+\right) \end{pmatrix},$$

and on the diagonal the i-th 2×2 block is

$$K^1_{ARW-s}\left(z_i, z_i\right) = \begin{pmatrix} 0 & c_t\left(z_i, z_i^+\right) - 1 \\ 1 - c_t\left(z_i, z_i^+\right) & 0 \end{pmatrix},$$

where

$$c_t\left(x, y\right) = \begin{cases} E_t\left(s_x s_y\right) & \text{if } x < y \\ -E_t\left(s_x s_y\right) & \text{if } x > y \\ 1 & \text{otherwise} \end{cases}.$$

where $x^+ = x + 1$.

The 2×1 *blocks are*

$$K^1_{ARW-s}(z_i, y_j) = \begin{pmatrix} \tilde{c}_t(z_i, y_j) \\ c_t(z_i^+, y_j) - c_t(z_i, y_j) \end{pmatrix},$$

where

$$\tilde{c}_t(x, y) = \begin{cases} E_t(s_x s_y) & \textit{if } x < y \\ -E_t(s_x s_y) & \textit{if } x > y \\ -1 & \textit{otherwise} \end{cases}.$$

The 1×1 *blocks are*

$$K^1_{ARW-s}(y_i, y_j) = c_{t_2}(y_i, y_j).$$

The blocks below the diagonal can be obtained by the identity (6.2)

Proof By defining the discrete derivative

$$\partial_{\xi_i} s_{\xi_i} = s_{\xi_i^+} - s_{\xi_i}$$

we have

$$n_{z_i} = \frac{1 - s_{z_i^+} s_{z_i}}{2} = \frac{-1}{2} s_{z_i}\left(s_{z_i^+} - s_{z_i}\right) = \left(\frac{-1}{2}\right) s_{z_i}\left(\partial_{\xi_i} s_{\xi_i}|_{\xi_i = z_i}\right)$$

where $\xi_i > z_i$.

Therefore,

$$\begin{aligned} \mathrm{E}\left[\prod_{i=1}^{N} n_{z_i} \prod_{j=1}^{2m} s_{y_j}\right] &= \left(\frac{-1}{2}\right)^N \mathrm{E}\left[\prod_{i=1}^{N} s_{z_i} \partial_{\xi_i} s_{\xi_i} \prod_{j=1}^{2m} s_{y_j}\right] |_{\xi_i = z_i} \\ &= \left(\frac{-1}{2}\right)^N \prod_{i=1}^{N} \partial_{\xi_i} \mathrm{E}\left[\prod_{i=1}^{N} s_{z_i} s_{\xi_i} \prod_{j=1}^{2m} s_{y_j}\right] |_{\xi_i = z_i} \end{aligned}$$

where $z_1 < \xi_1 < z_2 < \cdots < z_N$. The second equality is due to the face that both E and ∂_{ξ_i} are linear operators.

For this ordering $z_1 < \xi_1 < z_2 < \cdots < z_N < \xi_N < y_1 < \cdots < y_{2m}$ we know $\mathrm{E}\left[\prod_{i=1}^{N} s_{z_i} s_{\xi_i} \prod_{j=1}^{2m} s_{y_j}\right]$ is a $(2N + 2m) \times (2N + 2m)$ anti-symmetric matrix described in Theorem 5.

By Theorem 53 we know that

$$\mathrm{E}\left[\prod_{i=1}^{N} s_{z_i} s_{\xi_i} \prod_{j=1}^{2m} s_{y_j}\right] = \mathrm{Pf}\left[K_s^1\right].$$

The 2×2 blocks above the diagonal are in the first $2n$ row and columns. The i, j-th block is

$$K_s^1\left(z_i, \xi_i; z_j, \xi_j\right) = \begin{pmatrix} c_t\left(z_i, z_j\right) & c_t\left(z_i, \xi_j\right) \\ c_t\left(\xi_i, z_j\right) & c_t\left(\xi_i, \xi_j\right) \end{pmatrix}.$$

We can see the derivatives only apply to the second column and second row of the 2×2 block and they are functions of ξ_i and ξ_j. So after differentiating by ∂_{ξ_i} and ∂_{ξ_j} we have

$$\partial_{\xi_i}\partial_{\xi_j} K_s^1\left(z_i, \xi_i; z_j, \xi_j\right) = \begin{pmatrix} c\left(z_i, z_j\right) & c\left(z_i, \xi_j^+\right) - c\left(z_i, \xi_j\right) \\ c\left(\xi_i^+, z_j\right) - c\left(\xi_i, z_j\right) & 2c\left(\xi_i, \xi_j\right) - c\left(\xi_i^+, \xi_j\right) - c\left(\xi_i, \xi_j^+\right) \end{pmatrix}$$

where we set $c\left(\xi_i + 1, \xi_j + 1\right) = c\left(\xi_i, \xi_j\right)$ since the initial condition has translational symmetry.

The i-th 2×2 block on the diagonal is

$$K_s^1\left(z_i, \xi_i; z_i, \xi_i\right) = \begin{pmatrix} 0 & c\left(z_i, \xi_i\right) \\ -c\left(z_i, \xi_i\right) & 0 \end{pmatrix}.$$

and therefore only the derivative ∂_{ξ_i} applies to the block and after differentiating by ∂_{ξ_i} we get

$$\partial_{\xi_i} K_s^1\left(z_i, \xi_i; z_i, \xi_i\right) = \begin{pmatrix} 0 & c\left(z_i, \xi_i^+\right) - c\left(z_i, \xi_i\right) \\ -c\left(z_i, \xi_i^+\right) - c\left(z_i, \xi_i\right) & 0 \end{pmatrix}.$$

The 2×1 blocks over the diagonal are in the first $2n$ rows last $2m$ columns:

$$K_s^1\left(z_i, \xi_i; y_j\right) = \begin{pmatrix} \tilde{c}_t\left(z_i, y_j\right) \\ c_t\left(\xi_i, y_j\right) \end{pmatrix}$$

and after differentiating it by ∂_{ξ_i} we get

$$\partial_{\xi_i} K_s^1\left(z_i, \xi_i; y_j\right) = \begin{pmatrix} \tilde{c}_t\left(z_i, y_j\right) \\ c_t\left(\xi_i^+, y_j\right) - c_t\left(\xi_i, y_j\right) \end{pmatrix}$$

The 1×1 blocks are independent of ξ_i for all i so they act as constants and they are the same.

By setting $\xi_i = z_i$ for all i we have thus proved the theorem. $\square$

Theorem 22 is actually a special case of this theorem and we will generalise the theorem to the case of multi-time to prove the extended Pfaffian property of ARW.

6.2.2 2-time ARW-spin as PPP

After proving Theorem 54 we can easily generalise the result to multi-time. We start with two-time mixed correlation:

Theorem 55.
$$E\left[\prod_{i=1}^{N} n_{t_1,z_i}\prod_{j=1}^{2m} s_{t_2,y_j}\right] = \left(\frac{-1}{2}\right)^N Pf\left[K^2_{ARW-s}\right]$$

where the i,j-th block 2×2 blocks above the diagonal is

$$K^2_{ARW-s}\left(t_1,z_i;t_1,z_j\right) = \begin{pmatrix} c_{t_1}\left(z_i,z_j\right) & c_{t_1}\left(z_i,z_j^+\right) - c_{t_1}\left(z_i,z_j\right) \\ c_{t_1}\left(z_i^+,z_j\right) - c_{t_1}\left(z_i,z_j\right) & 2c_{t_1}\left(z_i,z_j\right) - c_{t_1}\left(z_i^+,z_j\right) - c_{t_1}\left(z_i,z_j^+\right) \end{pmatrix}$$

where $c_{t_1}(x,y)$ is the single-time spin correlation at time t_1.

The i-th block 2×2 blocks on the diagonal is:

$$K^2_{ARW-s}\left(t_1,z_i;t_1,z_i\right) = \begin{pmatrix} 0 & c_{t_1}\left(z_i,z_i^+\right) - 1 \\ 1 - c_{t_1}\left(z_i,z_i^+\right) & 0 \end{pmatrix}.$$

The 2×1 blocks over the diagonal are in the first $2n$ rows last $2m$ columns:

$$K^2_{ARW-s}\left(t_1,z_i;t_2,y_j\right) = \begin{pmatrix} \tilde{c}_{t_1,t_2}\left(z_i,y_j\right) \\ c_{t_1,t_2}\left(z_i^+,y_j\right) - c_{t_1,t_2}\left(z_i,y_j\right) \end{pmatrix}$$

where $c_{t_1,t_2}(x,y)$ is the two-time spin correlation which satisfies the kinetic equation in Lemma 46 and has the initial condition when $t_1 = t_2$ that

$$c_{t_1}(x,y) = \begin{cases} E_{t_1}\left(s_x s_y\right) & \text{if } x<y \\ -E_{t_1}\left(s_x s_y\right) & \text{if } x>y \\ 1 & \text{otherwise} \end{cases},$$

and $\tilde{c}_{t_1,t_2}(x,y)$*is similar except that it has a different initial condition when* $t_1 = t_2$

$$\tilde{c}_{t_1}(x,y) = \begin{cases} E_{t_1}(s_x s_y) & \textit{if } x < y \\ -E_{t_1}(s_x s_y) & \textit{if } x > y \\ -1 & \textit{otherwise} \end{cases}.$$

The 1×1 *blocks are*

$$K^2_{ARW-s}(t_2, y_i; t_2, y_j) = c_{t_2}(y_i, y_j).$$

Proof Since

$$n_{z_i} = \frac{1 - s_{z_i^+} s_{z_i}}{2},$$

the left hand side is just a linear combination of two-time paired spin correlation and thus satisfy the kinetic equation in Lemma 48, which is clearly the same kinetic equation satisfied by the right hand side because the entries of $\mathrm{Pf}\left[k^2_{ARW-s}\right]$ are two-time spin correlations $c_{t_1,t_2(x,y)}$ and $\tilde{c}_{t_1,t_2(x,y)}$ which satisfy the same kinetic equation for multi-time spin correlation. Therefore by Lemma 49 both sides satisfy the same kinetic equation.

When $s_{t_2,y_j} = s_{t_2,y_{j+1}}$, the left hand side will become a mixed correlation independent of s_{t_2,y_j} and $s_{t_2,y_{j+1}}$ because $s^2 = 1$. For the right hand side, the columns corresponding to s_{t_2,y_j} and $s_{t_2,y_{j+1}}$ are

$$\begin{matrix}
\cdots & \tilde{c}_{t_1,t_2}(z_1, y_j) & \tilde{c}_{t_1,t_2}(z_1, y_{j+1}) & \cdots \\
\cdots & c_{t_1,t_2}\left(z_1^+, y_j\right) - c_{t_1,t_2}(z_1, y_j) & c_{t_1,t_2}\left(z_1^+, y_{j+1}\right) - c_{t_1,t_2}(z_1, y_{j+1}) & \cdots \\
 & \vdots & \vdots & \\
\cdots & \tilde{c}_{t_1,t_2}(z_n, y_j) & \tilde{c}_{t_1,t_2}(z_n, y_{j+1}) & \cdots \\
\cdots & c_{t_1,t_2}\left(z_n^+, y_j\right) - c_{t_1,t_2}(z_n, y_j) & c_{t_1,t_2}\left(z_n^+, y_{j+1}\right) - c_{t_1,t_2}(z_n, y_{j+1}) & \cdots \\
\cdots & c_{t_1,t_2}(y_1, y_j) & c_{t_1,t_2}(y_1, y_{j+1}) & \cdots \\
 & \vdots & \vdots & \\
\cdots & c_{t_1,t_2}(y_{j-1}, y_j) & c_{t_1,t_2}(y_{j-1}, y_{j+1}) & \cdots \\
\cdots & \mathbf{0} & \mathbf{c_{t_1,t_2}(y_j, y_{j+1})} & \cdots \\
\cdots & -\mathbf{c_{t_1,t_2}(y_j, y_{j+1})} & \mathbf{0} & \cdots \\
\cdots & -c_{t_1,t_2}(y_j, y_{j+2}) & -c_{t_1,t_2}(y_{j+1}, y_{j+2}) & \cdots \\
 & \vdots & \vdots & \\
\cdots & -c_{t_1,t_2}(y_j, y_{2m}) & -c_{t_1,t_2}(y_{j+1}, y_{2m}) & \cdots
\end{matrix}.$$

When $s_{t_2,y_j} = s_{t_2,y_{j+1}}$, the two rows will become identical except for the entries in bold font and the same goes for the corresponding rows. The entries in bold font will become

$$\begin{array}{cccc} & \vdots & \vdots & \\ \cdots & 0 & 1 & \cdots \\ \cdots & -1 & 0 & \cdots \\ & \vdots & \vdots & \end{array}.$$

and hence by Lemma 52 the Pfaffian $\operatorname{Pf}\left[K^2_{ARW-s}\right]$ on the right hand side will be reduced to a Pfaffian by removing j-th and $j+1$-th column and row. So both sides satisfy the same boundary condition

As before in Theorem 54 by the identity

$$n_{z_i} = \left(\frac{-1}{2}\right) s_{z_i} \left(\partial_{\xi_i} s_{\xi_i}|_{\xi_i=z_i}\right)$$

we have

$$\begin{aligned} \mathrm{E}\left[\prod_{i=1}^{N} n_{t_1,z_i} \prod_{j=1}^{2m} s_{t_2,y_j}\right] &= \left(\frac{-1}{2}\right)^N \mathrm{E}\left[\prod_{i=1}^{N} s_{t_1,z_i} \partial_{\xi_i} s_{t_1,\xi_i} \prod_{j=1}^{2m} s_{t_2,y_j}\right] |_{\xi_i=z_i} \\ &= \left(\frac{-1}{2}\right)^N \prod_{i=1}^{N} \partial_{\xi_i} \mathrm{E}\left[\prod_{i=1}^{N} s_{t_1,z_i} s_{t_1,\xi_i} \prod_{j=1}^{2m} s_{t_2,y_j}\right] |_{\xi_i=z_i} \end{aligned}$$

where $z_1 < \xi_1 < z_2 < \cdots < z_N$.

From Theorem 53 we have

$$\mathrm{E}_t\left[\prod_{i=1}^{N} s_{t_1,z_i} s_{t_1,\xi_i} \prod_{j=1}^{2m} s_{t_1,y_j}\right] = \operatorname{Pf}\left[K^1_s\right].$$

Differentiating these blocks by $\prod_{i=1}^{N} \partial_{\xi_i}$.

The i-th 2×2 block on the diagonal of the first $2n \times 2n$ rows and columns is

$$\partial_{\xi_i} K^1_s\left(z_i, \xi_i; z_i, \xi_i\right) = \begin{pmatrix} 0 & c\left(z_i, \xi_i^+\right) - c\left(z_i, \xi_i\right) \\ -c\left(z_i, \xi_i^+\right) - c\left(z_i, \xi_i\right) & 0 \end{pmatrix}.$$

The i, j-th 2×2 block above the diagonal of the first $2n \times 2n$ rows and columns is

$$\partial_{\xi_i}\partial_{\xi_j}K_s^1\left(z_i,\xi_i;z_j,\xi_j\right)=\begin{pmatrix} c\left(z_i,z_j\right) & c\left(z_i,\xi_j^+\right)-c\left(z_i,\xi_j\right) \\ c\left(\xi_i^+,z_j\right)-c\left(\xi_i,z_j\right) & 2c\left(\xi_i,\xi_j\right)-c\left(\xi_i^+,\xi_j\right)-c\left(\xi_i,\xi_j^+\right) \end{pmatrix}.$$

The 2×1 blocks are in the first $2n$ rows and last $2m$ columns. They are:

$$\partial_{\xi_i}K_s^1\left(z_i,\xi_i;y_j\right)=\begin{pmatrix} \tilde{c}_t\left(z_i,y_j\right) \\ c_t\left(\xi_i^+,y_j\right)-c_t\left(\xi_i,y_j\right) \end{pmatrix}.$$

The 1×1 blocks are independent of ξ_i so we do not concern about them.

Setting $\xi_i = z_i$ for all i we have the blocks stated in the theorem and therefore the theorem is proved.

□

Now we can proceed to investigate k time ARW-spin correlation function.

6.2.3 k-time ARW-spin correlation

Theorem 56. *Suppose there are n ARW n_{t_i,z_i} existing in $k-1$ time slots $t_1 < t_2 < \cdots < t_{k-1}$ and $2m$ spins s_{t,y_j} existing at time $t > t_{k-1}$, then*

$$E\left[\prod_{i=1}^{n} n_{t_i,z_i}\prod_{j=1}^{2m} s_{t,y_j}\right]=\left(\frac{-1}{2}\right)^n Pf\left[K_{ARW-s}^k\right]$$

where K_{ARW-s}^k is a $(2n+2m)\times(2n+2m)$ anti-symmetric matrix with blocks. The i,j-th block 2×2 blocks above the diagonal is

$$\begin{aligned} &K_{ARW-s}^k\left(t_{z_i},z_i;t_{z_j},z_j\right)\\ =&\begin{pmatrix} c_{t_{z_i},t_{z_j}}\left(z_i,z_j\right) & c_{t_{z_i},t_{z_j}}\left(z_i,z_j^+\right)-c_{t_{z_i},t_{z_j}}\left(z_i,z_j\right) \\ c_{t_{z_i},t_{z_j}}\left(z_i^+,z_j\right)-c_{t_{z_i},t_{z_j}}\left(z_i,z_j\right) & 2c_{t_{z_i},t_{z_j}}\left(z_i,z_j\right)-c_{t_{z_i},t_{z_j}}\left(z_i^+,z_j\right)-c_{t_{z_i},t_{z_j}}\left(z_i,z_j^+\right) \end{pmatrix}\end{aligned}$$

where $c_{t_x,t_y}\left(x,y\right)$ is the multi-time spin correlation at the time t_x spin s_x exists and the time t_y spin s_y exists, which satisfies the kinetic equation in Lemma 46 and has the initial *condition when* $t_x = t_y$ *that*

$$c_{t_x}(x,y) = \begin{cases} E_{t_x}(s_x s_y) & \text{if } x < y \\ -E_{t_x}(s_x s_y) & \text{if } x > y \\ 1 & \text{otherwise} \end{cases}.$$

The i-th block 2×2 *blocks on the diagonal is:*

$$K^k_{ARW-s}(t_{z_i}, z_i; t_{z_i}, z_i) = \begin{pmatrix} 0 & c_{t_{z_i}}\left(z_i, z_i^+\right) - 1 \\ 1 - c_{t_{z_i}}\left(z_i, z_i^+\right) & 0 \end{pmatrix}$$

where $c_t(x,y)$ *is the single-time spin correlation function.*

The 2×1 *blocks over the diagonal are in the first* $2n$ *rows last* $2m$ *columns:*

$$K^k_{ARW-s}(t_{z_i}, z_i; t, y_j) = \begin{pmatrix} \tilde{c}_{t_{z_i},t}(z_i, y_j) \\ c_{t_{z_i},t}\left(z_i^+, y_j\right) - c_{t_{z_i},t}(z_i, y_j) \end{pmatrix}$$

where $\tilde{c}_{t_x,t_y}(x,y)$*is a function that satisfies the kinetic equation in Lemma 46 and has the initial condition that when* $t_x = t_y$

$$\tilde{c}_{t_x}(x,y) = \begin{cases} E_{t_x}(s_x s_y) & \text{if } x < y \\ -E_{t_x}(s_x s_y) & \text{if } x > y \\ -1 & \text{otherwise} \end{cases}.$$

The 1×1 *blocks are*

$$K^k_{ARW-s}(t, y_i; t, y_j) = c_t(y_i, y_j).$$

Proof Since

$$n_{z_i} = \frac{1 - s_{z_i^+} s_{z_i}}{2},$$

the left hand side is just a linear combination of two-time paired spin correlation and thus satisfy the kinetic equation in Lemma 48, which is clearly the same kinetic equation satisfied by the right hand side because the entries of $\mathrm{Pf}\left[k^2_{ARW-s}\right]$ all satisfy the same kinetic equation. Therefore, both sides satisfy the same kinetic equation.

We will prove both sides satisfy the same initial condition and boundary condition by induction. Suppose the theorem holds for $k-1$ time slots $t_1 < t_2 < \cdots < t_{k-2} < t$, that is, both sides are identical when $t = t_{k-2}$. Now we want to show that it also holds for k time slots $t_1 < t_2 < \cdots < t_{k-2} < t$, that is, both sides are identical when $t = t_{k-1}$.

When $t = t_{k-1}$, the left hand side becomes a new mixed ARW-spin correlation. There are ARW in the time slots $t_1, \ldots, t_{k-2}$ and a mixture of spins and ARW at time $t = t_{k-1}$. This is an object of which the form has not been determined yet. The goal is to show that it is represented by the Pfaffian $\mathrm{Pf}\left[K^k_{ARW-s}\right]$ at $t = t_{k-1}$.

The strategy is to use the identity

$$n_{z_i} = \left(\frac{-1}{2}\right) s_{z_i} \left(\partial_{\xi_i} s_{\xi_i}|_{\xi_i = z_i}\right)$$

to cast the new mixed ARW-spin correlation to the old mixed correlation.

Before we proceed let's re-label the ARW. At each time t_i, denote n_i the number of ARW at this time.

$$\begin{aligned}
& \mathrm{E}\left[\prod_{i=1}^{k-1}\prod_{i'=1}^{n_i} n_{t_i,z_{i'}} \prod_{j=1}^{2m} s_{t_k,y_j}\right] \\
= \; & \mathrm{E}\left[\prod_{i=1}^{k-2}\prod_{i'=1}^{n_i} n_{t_i,z_{i'}} \prod_{l=1}^{n_{k-1}} n_{t_{k-1},z_l} \prod_{j=1}^{2m} s_{t,y_j}\right] \\
= \; & \left(\frac{-1}{2}\right)^{n_{k-1}} \mathrm{E}\left[\prod_{i=1}^{k-2}\prod_{i'=1}^{n_i} n_{t_i,z_{i'}} \prod_{l=1}^{n_{k-1}} \left(s_{t_{k-1},z_l}\partial_{\xi_l} s_{t_{k-1},\xi_l}\right) \prod_{j=1}^{2m} s_{t,y_j}\right]|_{\xi_l = z_l} \\
= \; & \left(\frac{-1}{2}\right)^{n_{k-1}} \prod_{l=1}^{n_{k-1}} \partial_{\xi_i} \mathrm{E}\left[\prod_{i=1}^{k-2}\prod_{i'=1}^{n_i} n_{t_i,z_{i'}} \prod_{l=1}^{n_{k-1}} \left(s_{t_{k-1},z_l} s_{t_{k-1},\xi_l}\right) \prod_{j=1}^{2m} s_{t,y_j}\right]|_{\xi_l = z_l}
\end{aligned}$$

where the ordering $\cdots < z_i < \xi_i < z_{i+1} < \ldots$ is imposted.

By induction assumption the $k-1$-time mixed ARW-spin correlation

$$\mathrm{E}\left[\prod_{i=1}^{k-2}\prod_{i'=1}^{n_i} n_{t_i,z_{i'}} \prod_{l=1}^{n_{k-1}} \left(s_{t,z_l} s_{t,\xi_l}\right) \prod_{j=1}^{2m} s_{t,y_j}\right]$$

is almost given by the Pfaffian $\mathrm{Pf}\left[K^{k-1}_{ARW-s}\right]$.

The only difference is that the single-time spin correlations in the last $2m$ columns and last $2n_{k-1}$ rows from $2(n-n_{k-1})+1$-th row to $2n$-th row are of the form

$$\begin{matrix} & \vdots & \vdots & \\ \cdots & \tilde{c}_t\left(z_l, y_j\right) & \tilde{c}_t\left(z_l, y_{j+1}\right) & \cdots \\ \cdots & c_t\left(\xi_l, y_j\right) & c_t\left(\xi_l, y_{j+1}\right) & \cdots \\ & \vdots & \vdots & \end{matrix}$$

because we stipulate that no spin s_{t,y_j} is allowed between any pair of spins s_{t,z_l} and s_{t,ξ_l}.

The 2×2 blocks in the first $2(n-n_k)$ rows and columns of K^{k-1}_{ARW-s}, which corresponds to the multi-time correlation between the ARW $n_{t_i,z_{i'}}$ in the first $k-2$ time slots, are the same as that of K^k_{ARW-s} when $t=t_{k-1}$ and are independent of ξ_i for all i. The 2×1 blocks in the last $2m$ columns and first $2(n-n_{k-1})$ rows of K^{k-1}_{ARW-s}, which correspond to the multi-time correlation between spins s_{t,y_i} and the ARW $n_{t_i,z_{i'}}$ in the first $k-2$ time slots, are also the same as that of K^k_{ARW-s} when $t=t_{k-1}$ and are independent of ξ_i for all i. The 1×1 blocks in the last $2m$ columns and rows of K^{k-1}_{ARW-s}, which correspond to the multi-time correlation between spins s_{t,y_i} at $t=t_{k-1}$, are also the same as that of K^k_{ARW-s} when $t=t_{k-1}$ and are independent of ξ_i for all i.

The next step is to prove the blocks in three regions are equal to that of K^k_{ARW-s} when $t=t_{k-1}$ after taking the derivatives $\prod_{l=1}^{n_{k-1}}\partial_{\xi_i}$ and the limits $\xi_l=z_l$ for all $l=1,\ldots,n_{k-1}$.

The regions are:

1. from $2(n-n_{k-1})+1$-th column to $2n$-th column and in the first $2(n-n_{k-1})$ rows;
2. from $2(n-n_{k-1})+1$-th column to $2n$-th column and from $2(n-n_{k-1})+1$-th row to $2n$-th row;
3. from $2(n-n_{k-1})+1$-th row to $2n$-th row and in the last $2m$ columns.

In the first region we want to show the two 2×1 blocks will become a 2×2 block after taking the derivatives $\prod_{l=1}^{n_{k-1}}\partial_{\xi_i}$ and the limits $\xi_l=z_l$ for all $l=1,\ldots,n_{k-1}$.

Firstly consider the 2×1 block corresponding to s_{t_k,z_l} and n_{t_i,z_i}, which is

$$\begin{pmatrix} \tilde{c}_{t_i,t_k}\left(z_i, z_l\right) \\ c_{t_i,t_k}\left(z_i^+, z_l\right) - c_{t_i,t_k}\left(z_i, z_l\right) \end{pmatrix}.$$

As there is no dependence on ξ_l, taking the derivative ∂_{ξ_l} and taking the limit $\xi_l=z_i$ have no effect of the 2×1 block. In fact we can see this is exactly the first column of the 2×2 block of K^k_{ARW-s} when $t=t_{k-1}$.

Secondly consider the consider the 2×1 block corresponding to s_{t_k,ξ_l} and n_{t_i,z_i}, which is

$$\begin{pmatrix} \tilde{c}_{t_i,t_k}\left(z_i,\xi_l\right) \\ c_{t_i,t_k}\left(z_i^+,\xi_l\right)-c_{t_i,t_k}\left(z_i,\xi_l\right) \end{pmatrix}.$$

After taking the derivative ∂_{ξ_l} and taking the limit $\xi_l=z_i$ it becomes

$$\begin{pmatrix} \tilde{c}_{t_i,t_k}\left(z_i,z_l^+\right)-\tilde{c}_{t_i,t_k}\left(z_i,z_l\right) \\ 2c_{t_i,t_k}\left(z_i,z_l\right)-c_{t_i,t_k}\left(z_i^+,z_l\right)-c_{t_i,t_k}\left(z_i,z_l^+\right) \end{pmatrix},$$

which is exactly the second column of the 2×2 block of K^k_{ARW-s} when $t=t_{k-1}$.

Now consider the second region.

To recover the 2×2 blocks on the diagonal, we just have to notice that

$$\partial_{\xi_l}c_{t_k}\left(z_l,\xi_l\right)|_{\xi_l=z_l}=c_{t_k}\left(z_l,z_l^+\right)-1$$

which is exactly the nonzero terms in the 2×2 blocks on the diagonal of K^k_{ARW-s} when $t=t_{k-1}$.

To recover the 2×2 blocks above the diagonal we observe that

$$\begin{aligned}
&\partial_{\xi_i}\partial_{\xi_j}\begin{pmatrix} c_{t_{z_i},t_{z_j}}\left(z_i,z_j\right) & c_{t_{z_i},t_{z_j}}\left(z_i,\xi_j\right) \\ c_{t_{z_i},t_{z_j}}\left(\xi_i,z_j\right) & c_{t_{z_i},t_{z_j}}\left(\xi_i,\xi_j\right) \end{pmatrix}|_{\xi_i=z_i;\xi_j=z_j} \\
=&\begin{pmatrix} c_{t_{z_i},t_{z_j}}\left(z_i,z_j\right) & c_{t_{z_i},t_{z_j}}\left(z_i,z_j^+\right)-c_{t_{z_i},t_{z_j}}\left(z_i,z_j\right) \\ c_{t_{z_i},t_{z_j}}\left(z_i^+,z_j\right)-c_{t_{z_i},t_{z_j}}\left(z_i,z_j\right) & 2c_{t_{z_i},t_{z_j}}\left(z_i,z_j\right)-c_{t_{z_i},t_{z_j}}\left(z_i^+,z_j\right)-c_{t_{z_i},t_{z_j}}\left(z_i,z_j^+\right) \end{pmatrix}.
\end{aligned}$$

In the third region, observe that

$$\partial_{\xi_i}\begin{pmatrix} \tilde{c}_{t_i,t}\left(z_i,y_l\right) \\ c_{t_i,t}\left(\xi_i,y_l\right) \end{pmatrix}|_{\xi_i=z_i}=\begin{pmatrix} \tilde{c}_{t_{z_i},t}\left(z_i,y_l\right) \\ c_{t_{z_i},t}\left(z_i^+,y_l\right)-c_{t_{z_i},t}\left(z_i,y_l\right) \end{pmatrix}$$

To prove both sides satisfy the same boundary condition by induction, we assume the theorem holds for $k-1$ time slots and then consider the case of k time slots. Firstly we prove they satisfy the same reduction formula. When $y_j=y_{j+1}$, the columns corresponding to them are

$$\begin{array}{cccc}
\cdots & \tilde{c}_{t_{z_1},t}\left(z_1,y_j\right) & \tilde{c}_{t_{z_1},t}\left(z_1,y_{j+1}\right) & \cdots \\
\cdots & c_{t_{z_1},t}\left(z_1^+,y_j\right)-c_{t_{z_1},t}\left(z_1,y_j\right) & c_{t_{z_1},t}\left(z_1^+,y_{j+1}\right)-c_{t_{z_1},t}\left(z_1,y_{j+1}\right) & \cdots \\
 & \vdots & \vdots & \\
\cdots & \tilde{c}_{t_{z_n},t}\left(z_n,y_j\right) & \tilde{c}_{t_{z_n},t}\left(z_n,y_{j+1}\right) & \cdots \\
\cdots & c_{t_{z_n},t}\left(z_n^+,y_j\right)-c_{t_{z_n},t}\left(z_n,y_j\right) & c_{t_{z_n},t}\left(z_n^+,y_{j+1}\right)-c_{t_{z_n},t}\left(z_n,y_{j+1}\right) & \cdots \\
\cdots & c_t\left(y_1,y_j\right) & c_t\left(y_1,y_{j+1}\right) & \cdots \\
 & \vdots & \vdots & \\
\cdots & c_t\left(y_{j-1},y_j\right) & c_t\left(y_{j-1},y_{j+1}\right) & \cdots \\
\cdots & \mathbf{0} & \mathbf{c_t}\left(\mathbf{y_j},\mathbf{y_{j+1}}\right) & \cdots \\
\cdots & -\mathbf{c_t}\left(\mathbf{y_j},\mathbf{y_{j+1}}\right) & \mathbf{0} & \cdots \\
\cdots & -c_t\left(y_j,y_{j+2}\right) & -c_t\left(y_{j+1},y_{j+2}\right) & \cdots \\
 & \vdots & \vdots & \\
\cdots & -c_t\left(y_j,y_{2m}\right) & -c_t\left(y_{j+1},y_{2m}\right) & \cdots
\end{array}.$$

So when $y_j = y_{j+1}$ all the other entries are the same and the entries in bold font becomes

$$\begin{array}{cccc}
 & \vdots & \vdots & \\
\cdots & 0 & 1 & \cdots \\
\cdots & -1 & 0 & \cdots \\
 & \vdots & \vdots &
\end{array}$$

the same goes for the rows and hence by Lemma 52 we get the reduction formula we want. The left hand side has the same reduction formula because $s^2 = 1$.

Secondly we have to show that both sides reduce to the same formula when $2m = 0$.

After re-labeling the ARW's as before, we decompose the n's at time k by the same technique we used before:

$$
\begin{aligned}
& \mathrm{E}\left[\prod_{i=1}^{k-1}\prod_{i'=1}^{n_i} n_{t_i,z_{i'}}\right] \\
= \; & \mathrm{E}\left[\prod_{i=1}^{k-2}\prod_{i'=1}^{n_i} n_{t_i,z_{i'}} \prod_{j=1}^{n_{k-1}} n_{t_{k-1},z_j}\right] \\
= \; & \left(\frac{-1}{2}\right)^{n_{k-1}} \mathrm{E}\left[\prod_{i=1}^{k-2}\prod_{i'=1}^{n_i} n_{t_i,z_{i'}} \prod_{j=1}^{n_{k-1}} s_{t_{k-1},z_j}\partial_{\xi_j} s_{t_{k-1},\xi_j}\right] |_{\xi_j=z_j} \\
= \; & \left(\frac{-1}{2}\right)^{n_{k-1}} \prod_{j=1}^{n_{k-1}} \partial_{\xi_j} \mathrm{E}\left[\prod_{i=1}^{k-2}\prod_{i'=1}^{n_i} n_{t_i,z_{i'}} \prod_{j=1}^{n_{k-1}} s_{t_{k-1},z_j} s_{t_{k-1},\xi_j}\right] |_{\xi_j=z_j}.
\end{aligned}
$$

By exactly the same procedures of taking derivatives and limits as before we can prove that this is indeed equal to $K_{ARW-s}^{k-1}\left(t_{z_i}, z_i; t_{z_j}, z_j\right)$. Thus the theorem is proved

□

Now we are in position to prove the extended Pfaffian property of ARW.

Theorem 57.

$$
\rho^{ARW}\left(z_1, t_{z_1}; \ldots; z_n, t_{z_n}\right) = \left(\frac{-1}{2}\right) Pf\left[K_{ARW}^k\left(t_{z_i}, z_i; t_{z_j}, z_j\right)\right]
$$

where

$$
\begin{aligned}
& K_{ARW}^k\left(t_{z_i}, z_i; t_{z_j}, z_j\right) \\
= & \begin{pmatrix} c_{t_{z_i},t_{z_j}}\left(z_i, z_j\right) & c_{t_{z_i},t_{z_j}}\left(z_i, z_j^+\right) - c_{t_{z_i},t_{z_j}}\left(z_i, z_j\right) \\ c_{t_{z_i},t_{z_j}}\left(z_i^+, z_j\right) - c_{t_{z_i},t_{z_j}}\left(z_i, z_j\right) & 2c_{t_{z_i},t_{z_j}}\left(z_i, z_j\right) - c_{t_{z_i},t_{z_j}}\left(z_i^+, z_j\right) - c_{t_{z_i},t_{z_j}}\left(z_i, z_j^+\right) \end{pmatrix}
\end{aligned}
$$

over the diagonal and

$$
K_{ARW-s}^k\left(t_{z_i}, z_i; t_{z_i}, z_i\right) = \begin{pmatrix} 0 & c_{t_{z_i}}\left(z_i, z_i^+\right) - 1 \\ 1 - c_{t_{z_i}}\left(z_i, z_i^+\right) & 0 \end{pmatrix}
$$

on the diagonal and $c_{t_x,t_y}\left(x, y\right)$ *and* $c_{t_x}\left(x, x^+\right)$ *are defined in Theorem 56.*

Proof From Theorem 56 we can prove the theorem by letting $2m = 0$.

□

Remark By the Markov property of Glauber model, the multi-time spin correlation will also satisfy the kinetic equation (5.3). And since equation (5.3) is also of the form in Lemma 16, it can substitute in the proof and therefore the extended Pfaffian property also holds for the most general position-dependent ARW with creation of pairs of particles.

6.2.4 Alternative proof

We can prove Theorem 56 by another approach. Firstly we obtain the Pfaffian expression of k-time paired spin correlation. Then we apply the derivatives ∂_{ξ_i} and take the limit $\xi_i = z_i$ to obtain the multi-time mixed ARW-spin correlation.

k-time paired spin correlation

The gaol is to prove

$$\mathrm{E}\left[\prod_{i=1}^{n} s_{t_{z_i},z_i} s_{t_{z_i},\xi_i} \prod_{j=1}^{2m} s_{t,y_j}\right] = \mathrm{Pf}\left[K_s^k\right].$$

for a $(2n+2m)\times(2n+2m)$ anti-symmetric matrix K, where $z_1 < \xi_1 < z_2 < \cdots < z_n < \xi_n$ and $y_1 < \cdots < y_{2m}$ and no spin s_{t,y_j} at time t is allowed between any pairs of spins $s_{t_{z_i},z_i}$ and $s_{t_{z_i},\xi_i}$. Let us define the entries of the matrix here.

The 2×2 blocks are in the first $2n \times 2n$ rows and columns. On the diagonal they are:

$$K_s^k\left(t_{z_i}, z_i, \xi_i; t_{z_i}, z_i, \xi_i\right) = \begin{pmatrix} 0 & c_{t_{z_i}}(z_i, \xi_i) \\ -c_{t_{z_i}}(z_i, \xi_i) & 0 \end{pmatrix},$$

while above the diagonal the blocks are

$$K_s^k\left(t_{z_i}, z_i, \xi_i; t_{z_j}, z_j, \xi_j\right) = \begin{pmatrix} c_{t_{z_i},t_{z_j}}\left(z_i, z_j\right) & c_{t_{z_i},t_{z_j}}\left(z_i, \xi_j\right) \\ c_{t_{z_i},t_{z_j}}\left(\xi_j, z_j\right) & c_{t_{z_i},t_{z_j}}\left(\xi_i, \xi_j\right) \end{pmatrix}.$$

The 2×1 blocks are in the first $2n$ rows and last $2m$ columns. They are:

$$K_s^k\left(t_{z_i}, z_i, \xi_i; t, y_j\right) = \begin{pmatrix} \tilde{c}_{t_{z_i},t}\left(z_i, y_j\right) \\ c_{t_{z_i},t}\left(\xi_i, y_j\right) \end{pmatrix}.$$

where $c_{t_x,t_y}(x,y)$ is the 2-time spin correlation which satisfies the kinetic equation

$$\partial_{t_y} c_{t_x,t_y}(x,y) = \left[(-2\gamma D)\Delta_y + (-2D)(1+2\gamma)\right](x,y)$$

which was proved in Lemma 46 and has the initial condition that when $t_x = t_y$,

$$c_{t_x}(x,y) = \begin{cases} \mathrm{E}_{t_x}(s_x s_y) & \text{if } x < y \\ -\mathrm{E}_{t_x}(s_x s_y) & \text{if } x > y \\ 1 & \text{otherwise} \end{cases},$$

and $\tilde{c}_{t_x,t_y}(x,y)$is similar except that it has a different initial condition that when $t_x = t_y$,

$$\tilde{c}_{t_x}(x,y) = \begin{cases} \mathrm{E}_{t_x}(s_x s_y) & \text{if } x < y \\ -\mathrm{E}_{t_x}(s_x s_y) & \text{if } x > y \\ -1 & \text{otherwise} \end{cases}.$$

The 1×1 blocks are in the last $2m$ rows and last $2m$ columns. They are:

$$K_s^k(t, y_i; t, y_j) = c_t(y_i, y_j).$$

The blocks below the diagonal can be obtained by the identity (6.2).

All the steps are similar to those in the previous section. Firstly let us prove the lemmae that will be used in proving the initial condition.

Lemma 58. *When $y_i = z_j$ or $y_i = \xi_j$, where the spins $s_{t_{z_j}, z_j}$ and $s_{t_{z_j}, \xi_j}$ are at time t_k, the matrix whose Pfaffian is the multi-time paired spin correlation can be cast into the form at $t_k = t$:*

$$\begin{pmatrix} 0 & 1 & B \\ -1 & 0 & B \\ -B^T & -B^T & A \end{pmatrix}$$

where B is a $1 \times (2n + 2m - 2)$ row matrix and A is an anti-symmetric $(2n + 2m - 2) \times (2n + 2m - 2)$ matrix where independent of y_i and z_j or y_i and ξ_j.

Proof There are two cases to consider:

1. $z_j = y_i$
2. $\xi_j = y_i$

In the first case $z_j = y_i$ we observe that the columns corresponding to z_j and y_i respectively are

$$\begin{array}{cccccc}
\cdots & c_{t_{z_1},t_k}(z_1,z_j) & \cdots & \tilde{c}_{t_{z_1},t}(z_1,y_i) & \cdots \\
\cdots & c_{t_{z_1},t_k}(\xi_1,z_j) & \cdots & c_{t_{z_1},t}(\xi_1,y_i) & \cdots \\
 & \vdots & & \vdots & \\
\cdots & \mathbf{0} & \cdots & \mathbf{\tilde{c}_{t_{z_j},t}(z_j,y_i)} & \cdots \\
\cdots & -c_{t_k}(z_j,\xi_j) & \cdots & c_{t_{z_j},t}(\xi_j,y_i) & \cdots \\
 & \vdots & & \vdots & \\
\cdots & -c_{t_{z_n},t_k}(z_j,z_n) & \cdots & \tilde{c}_{t_{z_n},t}(z_n,y_i) & \cdots \\
\cdots & -c_{t_{z_n},t_k}(z_j,\xi_n) & \cdots & c_{t_{z_n},t}(\xi_n,y_i) & \cdots \\
\cdots & -\tilde{c}_{t_k,t}(z_j,y_1) & \cdots & c_t(y_1,y_i) & \cdots \\
 & \vdots & & \vdots & \\
\cdots & -\mathbf{\tilde{c}_{t_k,t}(z_j,y_i)} & \cdots & \mathbf{0} & \cdots \\
\cdots & -\tilde{c}_{t_k,t}(z_j,y_{i+1}) & \cdots & -c_t(y_i,y_{i+1}) & \cdots \\
 & \vdots & & \vdots & \\
\cdots & -\tilde{c}_{t_k,t}(z_j,y_{2m}) & \cdots & -c_t(y_i,y_{2m}) & \cdots
\end{array}.$$

When $z_j = y_i$ and $t_k = t$, the two columns will be identical except the entries in bold font which will become

$$\begin{array}{ccccc}
\cdots & 0 & \cdots & -1 & \cdots \\
 & \vdots & & \vdots & \\
\cdots & 1 & \cdots & 0 & \cdots
\end{array}.$$

By even permutation of row and column we can obtain the form stated in the lemma.

In the second case $\xi_j = y_i$ we observe that the columns corresponding to ξ_j and

y_i respectively are

$$\begin{array}{ccccc}
\cdots & c_{t_{z_1},t_k}(z_1,\xi_j) & \cdots & \tilde{c}_{t_{z_1},t}(z_1,y_i) & \cdots \\
\cdots & c_{t_{z_1},t_k}(\xi_1,\xi_j) & \cdots & c_{t_{z_1},t}(\xi_1,y_i) & \cdots \\
 & \vdots & & \vdots & \cdots \\
\cdots & c_{t_k}(z_j,\xi_j) & \cdots & \tilde{c}_{t_k,t}(z_j,y_i) & \cdots \\
\cdots & \mathbf{0} & \cdots & \mathbf{c_{t_k,t}}(\boldsymbol{\xi_j},\mathbf{y_i}) & \cdots \\
 & \vdots & & \vdots & \cdots \\
\cdots & -c_{t_{z_n},t_k}(\xi_j,z_n) & \cdots & \tilde{c}_{t_{z_n},t}(z_n,y_i) & \cdots \\
\cdots & -c_{t_{z_n},t_k}(\xi_j,\xi_n) & \cdots & c_{t_{z_n},t}(\xi_n,y_i) & \cdots \\
\cdots & -c_{t_k,t}(\xi_j,y_1) & \cdots & c_t(y_1,y_i) & \cdots \\
 & \vdots & & \vdots & \cdots \\
\cdots & \mathbf{c_{t_k,t}}(\boldsymbol{\xi_j},\mathbf{y_i}) & \cdots & \mathbf{0} & \cdots \\
\cdots & -c_{t_k,t}(\xi_j,y_{i+1}) & \cdots & -c_t(y_i,y_{i+1}) & \cdots \\
 & \vdots & & \vdots & \cdots \\
\cdots & -c_{t_k,t}(\xi_j,y_{2m}) & \cdots & -c_t(y_i,y_{2m}) & \cdots
\end{array}.$$

The two columns will be identical except the entries in bold font. The entries in the bold font will become

$$\begin{array}{ccccc}
\cdots & 0 & \cdots & 1 & \cdots \\
 & \vdots & & \vdots & . \\
\cdots & -1 & \cdots & 0 & \cdots
\end{array}$$

By even permutation of rows and columns we can obtain the form stated in the lemma again. □

Next we prove the lemma we will use to prove the boundary condition.

Lemma 59. *When $y_j = y_{j+1}$, $Pf\left[K_s^k\right]$ becomes*

$$Pf\begin{pmatrix} 0 & 1 & B \\ -1 & 0 & B \\ -B^T & -B^T & A \end{pmatrix}.$$

where A is a $(2n+2m-2)\times(2n+2m-2)$ anti-symmetric matrix independent of y_j and y_{j+1}.

Proof We observe that the columns corresponding to y_j and y_{j+1} respectively are

$$\begin{array}{cccc}
\cdots & \tilde{c}_{t_{z_1},t}\left(z_1, y_i\right) & \tilde{c}_{t_{z_1},t}\left(z_1, y_{i+1}\right) & \cdots \\
\cdots & c_{t_{z_1},t}\left(\xi_1, y_i\right) & c_{t_{z_1},t}\left(\xi_1, y_{i+1}\right) & \cdots \\
 & \vdots & \vdots & \\
\cdots & \tilde{c}_{t_{z_n},t}\left(z_n, y_i\right) & \tilde{c}_{t_{z_n},t}\left(z_n, y_{i+1}\right) & \cdots \\
\cdots & c_{t_{z_n},t}\left(\xi_n, y_i\right) & c_{t_{z_n},t}\left(\xi_n, y_{i+1}\right) & \cdots \\
\cdots & c_t\left(y_1, y_i\right) & c_t\left(y_1, y_{i+1}\right) & \cdots \\
 & \vdots & \vdots & \\
\cdots & c_t\left(y_{i-1}, y_i\right) & c_t\left(y_{i-1}, y_{i+1}\right) & \cdots \\
\cdots & \mathbf{0} & \mathbf{c_t}\left(\mathbf{y_i}, \mathbf{y_{i+1}}\right) & \cdots \\
\cdots & -\mathbf{c_t}\left(\mathbf{y_i}, \mathbf{y_{i+1}}\right) & \mathbf{0} & \cdots \\
\cdots & -c_t\left(y_i, y_{i+2}\right) & -c_t\left(y_{i+1}, y_{i+2}\right) & \cdots \\
 & \vdots & \vdots & \\
\cdots & -c_t\left(y_i, y_{2m}\right) & -c_t\left(y_{i+1}, y_{2m}\right) & \cdots
\end{array}.$$

When $y_j = y_{j+1}$, the two columns will be identical except for the entries in bold font which will become

$$\begin{array}{cccc}
 & \vdots & \vdots & \\
\cdots & 0 & 1 & \cdots \\
\cdots & -1 & 0 & \cdots \\
 & \vdots & \vdots &
\end{array}.$$

The same goes for the corresponding rows. Therefore by even row and column permutations we can obtain the matrix stated in the lemma. $\square$

Theorem 60.

$$E\left[\prod_{i=1}^{n} s_{t_1,z_i} s_{t_1,\xi_i} \prod_{j=1}^{2m} s_{t_2,y_j}\right] = Pf\left[K_s^k\right].$$

Proof From Lemma 48 and Lemma 49 we can see that both sides satisfy the same kinetic equation. And from Lemma 59 and Lemma 52 they satisfy the same boundary condition. From Lemma 58 and Lemma 52 they satisfy the same initial condition. Therefore by the uniqueness Theorem 31 they are identical. $\square$

By Theorem 60 we can then derive the multi-time mixed ARW-spin correlation by the procedures:

$$
\begin{aligned}
& \mathrm{E}\left[\prod_{i=1}^{n} n_{t_i,z_i} \prod_{j=1}^{2m} s_{t,y_j}\right] \\
= & \left(\frac{-1}{2}\right)^n \mathrm{E}\left[\prod_{i=1}^{n} \left(s_{t_i,z_i} \partial_{\xi_i} s_{t_i,\xi_i}\right) \prod_{j=1}^{2m} s_{t,y_j}\right] |_{\xi_i=z_i} \\
= & \left(\frac{-1}{2}\right)^n \prod_{l=1}^{n} \partial_{\xi_i} \mathrm{E}\left[\prod_{i=1}^{n} \left(s_{t_i,z_i} s_{t_i,\xi_i}\right) \prod_{j=1}^{2m} s_{t,y_j}\right] |_{\xi_i=z_i} \\
= & \left(\frac{-1}{2}\right)^n \mathrm{Pf}\left[K^k_{ARW-s}\right].
\end{aligned}
$$

Therefore we can recover the result in Theorem 56.

6.2.5 Third proof

By using Lemma 1 we can also have a less general proof that $\mathrm{E}\left[\prod_{i=1}^{n} n_{t_i,z_i}\right]$ has extended Pfaffian property.

Observe that

$$
\begin{aligned}
& \mathrm{E}\left[\prod_{i=1}^{n} n_{t_i,z_i}\right] \\
= & \left(\frac{1}{2}\right)^n \mathrm{E}\left[\prod_{i=1}^{n} 1 - s_{t_i,z_i} s_{t_i,z_i^+}\right] \\
= & \left(\frac{1}{2}\right)^n \mathrm{E}\left[1 - \sum_{i=1}^{n} s_{t_i,z_i} s_{t_i,z_i^+} + \sum_{i<j} s_{t_i,z_i} s_{t_i,z_i^+} s_{t_j,z_j} s_{t_j,z_j^+} + \cdots + (-1)^n \prod_{i=1}^{n} s_{t_i,z_i} s_{t_i,z_i^+}\right]
\end{aligned}
$$

By Theorem 60 we know that the paired spin correlations can be expressed as a Pfaffian whose entries are the 2×2 blocks of $\mathrm{Pf}\left[K^k_s\right]$. Denote the special case of K^k_s by $\tilde{K}^k_s$ So we have

$$
\begin{aligned}
& \mathrm{E}\left[\prod_{i=1}^{n} n_{t_i,z_i}\right] \\
= & \left(\frac{-1}{2}\right)^n 1 - \sum_{J_2} \mathrm{Pf}\left(\tilde{K}^k_s|_{J_2}\right) + \sum_{J_4} \mathrm{Pf}\left(\tilde{K}^k_s|_{J_4}\right) + \cdots + (-1)^n \mathrm{Pf}\left(\tilde{K}^k_s\right) \\
= & \mathrm{Pf}\left[I - \tilde{K}^k_s\right]
\end{aligned}
$$

where I is a $2n \times 2n$ block diagonal matrix with 2×2 blocks $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on the diagonal and the last equality comes from Lemma 1. The Pfaffian Pf $\left[I - \tilde{K_s^k}\right]$ obtained here is equivalent to the one obtained by the previous two methods. This can be seen from the remark in Section 4.1.1.

6.3 Multi-time correlation of CRW

Since CRW and ARW have similar correlation function we might expect CRW also has extended Pfaffian property , at least in the case that there is no immigration of particles. However, this is not true. Firstly we can show that the multi-time interval probability satisfy the kinetic equation:

$$\begin{aligned} & \partial_t P\left(\Omega_{x_1,y_1:t_{x_1}} \cap \cdots \cap \Omega_{x_n,y_n;t_{x_n}} \cap \Omega_{z_1,\xi_1;t} \cap \Omega_{z_m,\xi_m;t}\right) \\ = \quad & D\tilde{\Delta} P\left(\Omega_{x_1,y_1:t_{x_1}} \cap \cdots \cap \Omega_{x_n,y_n;t_{x_n}} \cap \Omega_{z_1,\xi_1;t} \cap \Omega_{z_m,\xi_m;t}\right) \end{aligned}$$

where $\tilde{\Delta} = \sum_{i=1}^{n} \left(\tilde{\Delta}_{z_i} + \tilde{\Delta}_{\xi_i}\right)$ and

$$\tilde{\Delta}_{x_i} f\left(x_i\right) = 2p_{x_i} f\left(x_i + 1\right) - 2f\left(x_i\right) + 2\left(1 - p_{x_i}\right) f\left(x_i - 1\right).$$

It satisfies exactly the same equation as that in Section 5.5.2 because of the Markov property of the system. We can see that the kinetic equation is also equal to that of multi-time spin correlation in Lemma 50 for zero temperature, which corresponds to $\gamma = \frac{-1}{2}$.

However, the multi-time CRW does not preserve extended Pfaffian property because the initial conditions do not match.

Consider the 4pt case which has 4 coordinates

$$(x_1, t_1), (y_1, t_1), (x_2, t_2) \text{ and } (y_2, t_2)$$

with the condition

$$x_1 < x_2 < y_1 < y_2.$$

Although we can prove the multi-time empty interval probability and the multi-time spin correlation satisfy the same kinetic equation, we can show that they do not have the same initial condition and therefore are not identical functions. Consider the initial condition $t_1 = t_2$.

Now at time $t_1 = t_2$, on one hand the spin correlation is still a function dependent on x_2 and y_1 while on the other hand the empty interval probability will be independent of x_2 and y_1. Therefore they do not satisfy the same initial condition.

As the empty interval probability and the paired spin correlation are not identical, we cannot imitate the construction of the ARW case in section 6.2 and therefore the multi-time CRW is not likely to have extended Pfaffian property.

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